DOI: 10.5281/zenodo.15097702
Link: https://zenodo.org/records/15097702

ON THE ESSENTIAL SPECTRUM OF A MATRIX OPERATOR ON A HILBERT SPACE

Shukhrat Alladustov, DSc.
Department of Higher and Applied Mathematics,
Tashkent State University of Economics,
E-mail: sh.alladustov@gmail.com, Tel: +998979171121

Abstract. *We study essential spectrum of a matrix operator $\mathbb{H}$, that describes three particles interacting in the direct sum of certain subspaces of the Fock space. It is shown that the essential spectrum of this operator lies in the real axis and is described as a union of segments. Moreover, we establish the maximum for number of segments.*
Keywords: *Schrödinger operator, Fock space, channel operators, spectrum, eigenvalue.*

I. INTRODUCTION

In statistical physics [1,2], in solid-state physics [3,4] and in the theory of quantum fields [5,6], studying the spectral properties of the operators associated with a system of three particles is one of the intensive research areas. Sigal *et al* [7] found the location of the spectrum for a system of several particles. They showed that continuous spectrum does not contain singular points by using some geometric techniques. They also found the regions where points of the discrete spectrum accumulate.

The study of systems describing n particles can be reduced to the study of self-adjoint operators acting in the so-called cut subspace $H^{(n)}$ of the Fock space, consisting of $r \le n$ particles (see the references [1,4,5,6,7,8]). The model operators, associated to a system of two bosons and another particle of a different nature were studied by Albeverio $et\ al$ [9,10] in the subspaces of the bosonic Fock space. They explicitly described the essential spectrum and found its location. In this work, we study the essential spectrum of a similar model operator but in the fermionic Fock space. We find the exact location of the essential spectrum under certain conditions. We also show that the essential spectrum can be described by the spectrum of the operator $h(p), p \in T^3$, called a Friedreich model. In the main theorem of the paper (Theorem 3), we obtain that the essential spectrum of the operator $\mathbb{H}$ consists of a union of closed intervals, and the number of intervals does not exceed four.

The paper is organized as follows. Section I is the introduction. In Section II, we describe the model operator, $\mathbb{H}$. In Section III, channel operators and Friedrichs models are defined, and their properties are given. The main results, including their proofs, are presented in Section IV.

II. THE MODEL OPERATOR

We use the following notations:

$\mathbb{T}^3 = (-\pi, \pi]^3$ – the three-dimensional torus.

$L^2((\mathbb{T}^3)^2)$ – the Hilbert space of square-integrable functions in $(\mathbb{T}^3)^2$.

$L^2_{as}((\mathbb{T}^3)^2)$ – the subspace of antisymmetric functions.

For simplicity, denote

$$\mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_1 = L^2(\mathbb{T}^3), \quad \mathcal{H}_2 = L^2_{as}((\mathbb{T}^3)^2).$$

Let I_j and $\langle\cdot,\cdot\rangle_j$, $j = 0,1,2$ be an identity operator and an inner product in these spaces, respectively. Then, we denote their direct sum as

$$\boldsymbol{\mathcal{H}} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2.$$

For any functions $g \in L^2(\mathbb{T}^3)$, we define the corresponding operators $\boldsymbol{g}: \mathcal{H}_1 \to \mathcal{H}_0$ and $\boldsymbol{g}^*: \mathcal{H}_0 \to \mathcal{H}_1$ as

$$\boldsymbol{g}(f) = \langle f, g\rangle_1, f \in \mathcal{H}_1$$

and

$$\boldsymbol{g}^*(c) = c(\cdot)g, \ c \in \mathcal{H}_0,$$

respectively.

Let $L_g, L_{sg}: L^2((\mathbb{T}^3)^2) \to L^2(\mathbb{T}^3)$ and $L_g^*, L_{sg}^*: L^2(\mathbb{T}^3) \to L^2((\mathbb{T}^3)^2)$ be operators defined as

$$L_g = I_1 \otimes \boldsymbol{g}, \quad L_{sg} = \boldsymbol{g} \otimes I_1$$

and

$$L_g^* = I_1 \otimes \boldsymbol{g}^*, \quad L_{sg}^* = \boldsymbol{g}^* \otimes I_1,$$

respectively.

Next, we define the model operator as a matrix operator in the Hilbert space $\boldsymbol{\mathcal{H}}$ as

$$\mathbb{H} = \begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{10} & H_{11} & H_{12} \\ 0 & H_{21} & H_{22} \end{pmatrix},$$

where the operators $H_{i,j}$ act as follows:

$$H_{00}: \mathbb{C} \to \mathbb{C}, \ H_{00}(c) = u_0 c$$
$$H_{01}: L^2(\mathbb{T}^3) \to \mathbb{C}, \ H_{01}(f) = \boldsymbol{a}(f)$$
$$H_{10}: \mathbb{C} \to L^2(\mathbb{T}^3), \ H_{10}(c) = c \cdot a$$
$$H_{11}: L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3), \ H_{11}f(p) = u(p)f(p) + \int_{\mathbb{T}^3} w(p,t)f(t)dt$$
$$H_{12}: L^2((\mathbb{T}^3)^2) \to L^2(\mathbb{T}^3), \qquad H_{12}(\varphi) = L_b(\varphi)$$
$$H_{21}: L^2(\mathbb{T}^3) \to L^2((\mathbb{T}^3)^2), \qquad H_{21}(f) = \frac{1}{2}(L_b^* - L_{sb}^*)(f)$$
$$H_{22}: L^2((\mathbb{T}^3)^2) \to L^2((\mathbb{T}^3)^2),$$
$$H_{22}(\varphi)(p,q) = \varphi(p,q)E(p,q) - (L_\varphi^* L_\varphi - L_{s\varphi}^* L_{s\varphi})(\varphi)(p,q)$$

Here u_0 is a fixed number, $a, \ b, \ u, \ \varphi$ are real-valued continuous functions on the torus $\mathbb{T}^3$, $E(\cdot,\cdot)$ is a real-valued continuous symmetric function in $(\mathbb{T}^3)^2$, and $w(\cdot,\cdot) \in L^2((\mathbb{T}^3)^2)$ is a self-adjoint function, i.e., $w(p,q) = \overline{w(q,p)}$, $p, q \in \mathbb{T}^3$. We note that with these definitions, H_{01} and H_{12} are annihilation operators, which lowers the number of particles in a given state by one, while H_{10} and H_{21} are creation operators, being adjoint to the annihilation operators, increase the number of particles by one (see Ref. [5] for more details).

Defined in this way, $\mathbb{H}$ is a bounded and self-adjoint operator in $\boldsymbol{\mathcal{H}}$.

III. CHANNEL OPERATORS AND A FRIEDRICHS MODEL

Here, we define an operator H_{ch}, which acts in $\widehat{\mathcal{H}} = L^2(\mathbb{T}^3) \oplus L^2((\mathbb{T}^3)^2)$ as a matrix operator

$$H_{ch} = \begin{pmatrix} H_{11} & \frac{1}{\sqrt{2}} L_b \\ \frac{1}{\sqrt{2}} L_b^* & H_{22}^0 - L_\varphi^* L_\varphi \end{pmatrix}.$$

This operator is called *a channel operator* corresponding to $\mathbb{H}$ (see, Ref. [11]). We can see that H_{ch} has a simpler structure than the model operator, $\mathbb{H}$. Therefore, description of the spectral properties of the channel operator is also much easier. There is a particular relation between the essential spectra of the channel operators and the model operator $\mathbb{H}$, which will be given later.

Let U_α be a multiplication operator in $L^2(\mathbb{T}^3)$, by the function $\alpha(\cdot)$, i.e.,

$$U_\alpha \begin{pmatrix} f_1(p) \\ f_2(p,q) \end{pmatrix} = \begin{pmatrix} \alpha(p) f_1(p) \\ \alpha(p) f_2(p,q) \end{pmatrix}, \quad \alpha \in L^2(\mathbb{T}^3).$$

We note that H_{ch} commutes with the abelian group of all operators U_α, i.e., for any $\alpha \in L^2(\mathbb{T}^3)$, we have

$$H_{ch} U_\alpha = U_\alpha H_{ch}.$$

The channel operator H_{ch} can be decomposed into a direct von Neumann integral (see e.g., [12, Theorem XIII.84])

$$H_{ch} = \int_{\mathbb{T}^3} \oplus H(p) dp. \tag{1}$$

Here $H(p)$, $p \in \mathbb{T}^3$, the so called Friedrichs model, is defined as a bounded and self-adjoint operator acting in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$H(p) = H_0(p) + V,$$

where

$$H_0(p) = \begin{pmatrix} 0 & 0 \\ 0 & h_0(p) \end{pmatrix}, \quad V = \begin{pmatrix} u(p) & \frac{1}{\sqrt{2}} \boldsymbol{b} \\ \frac{1}{\sqrt{2}} \boldsymbol{b}^* & -\varphi^* \varphi \end{pmatrix},$$

with $h_0(p)$, $p \in \mathbb{T}^3$ being a multiplication operator by the function $e_p(\cdot) := E(p,\cdot)$,

$$(h_0(p) f)(q) = e_p(q) f(q), \quad f \in \mathcal{H}_1.$$

For spectral properties of such types of Friedrichs models refer to Refs. [9,10].

From the decomposition (1) and the theorem on the spectrum of decomposable operators ([12, Theorem XIII.85]), we obtain the following theorem.

Theorem 1. The spectrum of the channel operator H_{ch} is described as

$$\sigma(H_{ch}) = \cup_{p \in \mathbb{T}^3} \{\sigma_d(H(p))\} \cup [E_{min}, E_{max}]$$

where $\sigma_d(H(p))$ is the discrete spectrum of the operator $H(p), p \in \mathbb{T}^3$ and

$$E_{\min} = \min_{p,q \in \mathbb{T}^3} E(p,q), \qquad E_{\max} = \max_{p,q \in \mathbb{T}^3} E(p,q)$$

Next, we discuss spectral properties of the Friedrichs model $H(p)$, $p \in \mathbb{T}^3$. According to the definition of the Friedrichs model, $H(p) = H_0(p) + V$, the perturbation operator V is an operator of finite rank. Therefore, from Weyl's theorem on the stability of the spectra, $\sigma_{ess}(H(p))$, the essential spectrum of the operator $H(p)$, coincides with the spectrum of $h_0(p)$, i.e.,

$$\sigma_{ess}(H(p)) = \sigma\big(h_0(p)\big) = [m(p), M(p)], \quad p \in \mathbb{T}^3,$$

where $m(p)$ and $M(p)$ are the maximum and minimum values of the function $e_p(q)$ in the torus $\mathbb{T}^3$, i.e.,

$$m(p) = \min_{q\in\mathbb{T}^3} e_p(q)\,, \;\; M(p) = \max_{q\in\mathbb{T}^3} e_p(q)$$

Next, for any $p \in \mathbb{T}^3$, we define $\Delta(p,\cdot)$, the Fredholm determinant associated with the operator $H(p)$, as an analytic function in $\mathbb{C}\backslash[m(p), M(p)]$ by

$$\Delta(p;z) = (I_0 - \varphi r_0(p,z)\varphi^*)\left(H_{00}(p) - zI_0 - \frac{1}{2}\boldsymbol{b}r_0(p,z)\boldsymbol{b}^*\right) - \frac{1}{2}(\boldsymbol{b}r_0(p,z)\varphi^*)^2$$

where $r_0(p;z)$, $z \in \mathbb{C}\backslash[m(p), M(p)]$, is the resolvent of $h_0(p), p \in \mathbb{T}^3$,

$$r_0(p;z)(\varphi) = \int_{\mathbb{T}^3} \frac{\varphi(s)ds}{E(p,s) - z}$$

Explicitly, $\Delta(p;z)$ can be written as

$$\Delta(p;z) = \left(1 - \int_{\mathbb{T}^3} \frac{\varphi^2(s)ds}{E(p,s)-z}\right)\left(u(p) - z - \frac{1}{2}\int_{\mathbb{T}^3} \frac{b^2(s)ds}{E(p,s)-z}\right) - \frac{1}{2}\left(\int_{\mathbb{T}^3} \frac{b(s)\varphi(s)ds}{E(p,s)-z}\right)^2.$$

Lemma 1. For any $p \in \mathbb{T}^3$, the number $z \in \mathbb{C}\backslash[m(p), M(p)]$ is an eigenvalue of $H(p)$ if and only if $\Delta(p,z) = 0$.

Proof. According to the definition of the operator $H(p)$, the eigenvalue equation,

$$H(p)f = zf, \quad f \in \mathcal{H}^{(2)}, \tag{2}$$

is equivalent to the system of equations,

$$\begin{cases} (u(p) - z)f_0 + \frac{1}{\sqrt{2}}\boldsymbol{b}f_1 = 0 \\ -\frac{1}{\sqrt{2}}r_0(p;z)\boldsymbol{b}^*f_0 + r_0(p;z)\varphi^*\varphi f_1 = f_1 \end{cases}, \; f = (f_0, f_1) \in \mathcal{H}^{(2)}.$$

In other words, we have

$$\begin{cases} (H_{00}(p) - zI_0 - \frac{1}{2}\boldsymbol{b}r_0(p,z)\boldsymbol{b}^*)f_0 & + \frac{1}{\sqrt{2}}\boldsymbol{b}r_0(p,z)\varphi^*\alpha = 0 \\ -\frac{1}{\sqrt{2}}\varphi r_0(p,z)\boldsymbol{b}^*f_0 & +(I_0 - \varphi r_0(p,z)\varphi^*)\alpha = 0 \end{cases}, \tag{3}$$

where $f_0, \alpha \in \mathbb{C}$.

Solutions of the last system of linear equations and the equation (2) are related as

$$f_0 = f_0, \alpha = \varphi f_1$$

and

$$f_1(q) = r_0(p;z)\left(-\frac{1}{2}b(q)f_0 + \varphi(q)\alpha\right).$$

On the other hand, determinant of the system of equation (3) is equal to $\Delta(p;z)$, therefore the equation $H(p)f = zf, f \in \mathcal{H}_0 \oplus \mathcal{H}_1$ has a nontrivial solution if and only if $\Delta(p;z) = 0$. This completes the proof of the lemma.

Denote the number of eigenvalues of $\mathbb{H}$ lying below z $\big(z \leq inf\ \sigma_{\text{ess}}(A)\big)$, by $n_-(A,z)$ and the number of eigenvalues lying above z $\big(z \geq sup\ \sigma_{\text{ess}}(A)\big)$ by $n_+(A,z)$, counted with multiplicities.

Lemma 2. For a fixed $p \in \mathbb{T}^3$, we have:

$n_-\big(H(p), m(p)\big) \leq 1$ if $\varphi(\cdot)$ and $b(\cdot)$ are bounded operators

$n_-\big(H(p), m(p)\big) \leq 2$, if they are unbounded.

Moreover, we have

$$n_+\big(H(p), M(p)\big) \le 1.$$

Proof. We can easily obtain that

$$z_{1,2} = -\frac{u(p) \pm \sqrt{u^2(p) + 4||b||^2}}{2}, \qquad z_1 < 0 < z_2$$

are simple eigenvalues of the matrix operator

$$V' = \begin{pmatrix} u(p) & \frac{1}{\sqrt{2}}\boldsymbol{b} \\ \frac{1}{\sqrt{2}}\boldsymbol{b}^* & 0 \end{pmatrix}.$$

From the definition of the operator V, $\mathrm{Img}(V) = \mathbb{C} \bigoplus \langle b, \varphi \rangle$, where $\langle b, \varphi \rangle$ is the subspace spanned by b and φ. Therefore, according to the facts that $-\varphi^*\varphi \le 0$ and $\sigma_{ess}(V') = \{0\}$, the following statements hold true for the operator V:

i) If the functions $\varphi(\cdot)$ and $b(\cdot)$ are linearly bound, then V has two positive eigenvalues (with multiplicities) and one (simple) negative eigenvalue.

ii) If the functions $\varphi(\cdot)$ and $b(\cdot)$ are linearly unbounded, then V has only one (simple) positive and one (simple) negative eigenvalues.

Also, from the relations $H(p) \ge m(p) + V$ and $H(p) \le M(p) + V$, and the minimax principle, for $n_-\big(H(p), m(p)\big)$ and $n_+\big(H(p), M(p)\big)$, we have

$$n_-\big(H(p), m(p)\big) \le n_-\big(m(p) + V, m(p)\big),$$
$$n_+(H(p), M(p)) \le n_+\big(M(p) + V, M(p)\big),$$

respectively. Then, the assertions **i)** and **ii)**, as well as the relations

$$n_-\big(m(p) + V, m(p)\big) = n_-(V, 0)$$

and

$$n_+\big(M(p) + V, M(p)\big) = n_+(V, 0)$$

yields the proof. The following can be derived from this lemma.

Corollary 1. The Fredholm operator $\Delta(p,\cdot), p \in \mathbb{T}^3$, may have no more than one zero (two zeros) in the interval $\big(-\infty, m(p)\big)$, if $\varphi(\cdot)$ and $b(\cdot)$ are linearly bounded (resp. unbounded). Furthermore, it may have only one zero in the interval $(M(p), \infty)$.

Let $\Sigma = \sigma(H_{ch})$ be the spectrum of the channel operator H_{ch}, then we have

$$\Sigma = [E_{min}, E_{max}] \cup \sigma_2$$

where $\sigma_2 = \cup_{p \in \mathbb{T}^3} \sigma_d\big(H(p)\big)$, i.e.,

$$\sigma_2 = \{z \in \mathbb{R}\backslash[m(p), M(p)] : \Delta(p; z) = 0 \text{ for some } p \in \mathbb{T}^3\}.$$

Theorem 2. The essential spectrum, $\sigma_{ess}(\mathbb{H})$, of the model operator $\mathbb{H}$ coincides with the set Σ, i.e.,

$$\sigma_{ess}(\mathbb{H}) = \Sigma.$$

Proof. The proof can be found in Ref. [9].

IV. FORMULATION AND PROOF OF THE MAIN RESULTS

Here we present main results of the paper, which describes $\sigma_{ess}(\mathbb{H})$, the essential spectrum of the model operator $\mathbb{H}$.

Theorem 3. $\sigma_{ess}(\mathbb{H})$ consists of a union of no more than four closed intervals.

Proof. Let $\sigma \subset \sigma(\mathbb{H})$ be the largest closed interval containing $[E_{min}, E_{max}]$, which may coincide with $\sigma = [E_{min}, E_{max}]$. Set

$$\sigma_0 := \cup_{p\in\mathbb{T}^3} \{\sigma_d(H(p))\}\backslash\sigma. \tag{4}$$

If σ_0 is an empty set, then the essential spectrum, $\sigma_{ess}(\mathbb{H})$, consists of only one closed interval σ.

If σ_0 is not an empty set, then $\sigma_0 \cap \sigma = \emptyset$ and from the definition of a spectrum the set σ_0 is closed.

According to the definition (4), for any $p \in \mathbb{T}^3$, the operator $H(p)$ has an eigenvalue in σ_0.

Next, let the inclusion $[a, b] \subset \sigma_0$ hold, where a, b lie in the boundary of $\sigma(\mathbb{H})$. Let G_ω be a set of points $p \in \mathbb{T}^3$, such that $H(p)$ has an eigenvalue in $[a, b]$. We show that $G_\omega = \mathbb{T}^3$. Let $p_0 \in G_\omega$. Then, due to Theorem 2 and Lemma 1, there exists a number $z_0 \in [a, b]$, such that $\Delta(p_0, z_0) = 0$.

However, for any $p \in \mathbb{T}^3$, the function $\Delta(p,\cdot)$ is analytic in some region containing the interval$[a, b]$. Therefore,

$$\frac{\partial^1}{\partial z^1}\Delta(p_0, z_0) \neq 0$$

or otherwise

$$\frac{\partial^2}{\partial z^2}\Delta(p_0, z_0) \neq 0.$$

According to the implicit function theorem there exist neighborhoods $U(p_0) \subset \mathbb{T}^3$ and $U'(z_0) \subset [a, b]$ of the points p_0 and z_0, respectively, and a continuous function

$$z: U(p_0) \to U'(z_0)$$

such that $\Delta(p, z(p)) \equiv 0$ for all $p \in U(p_0)$.

By Lemma 1, the number $z(p) \in [a, b]$ is an eigenvalue of the operator $H(p)$ for any $p \in U(p_0) \subset G_\omega$, which yields that G_ω is an open set.

Next, we prove the closedness of the set G_ω. Indeed, let a sequence $\{p_n\} \subset G_\omega$ converge to $p_0 \in \mathbb{T}^3$ and let $\{z(p_n)\} \subset [a, b]$ be an eigenvalue of the operator $H(p_n)$.

Without loss of a generality we may assume that

$$\lim_{n\to\infty} z(p_n) = z_0 \in [a, b].$$

As $\Delta(\cdot,\cdot)$ is a continuous function in $\mathbb{T}^3 \times [a, b]$, we have

$$0 \equiv \lim_{n\to\infty} \Delta(p_n, z(p_n)) = \Delta(p_0, z_0)$$

and therefore $p_0 \in G_\omega$ since $[a, b]$ is closed. Hence, the set G_ω is closed. So, we showed that the set G_ω is both open and closed, and therefore we conclude $G_\omega = \mathbb{T}^3$.

According to Corollary 1, the $d(H(p))$, $p \in \mathbb{T}^3$ contains no more than three eigenvalues, therefore the number of closed intervals $[a, b] \subset \sigma_0$ does not exceed three.

From the relation,

$$\sigma_{ess}(H) = \sigma_0 \cup [E_{min}, E_{max}],$$

we conclude that the essential spectrum $\sigma_{ess}(\mathbb{H})$ consists of a union of no more than four segments.

V. REFERENCES

1. R.A. Minlos, H. Spohn, The three-body problem in radioactive decay: the case of one atom and at most two photons // Topics in Statistical and Theoretical Physics. Amer. Math. Soc. Transl. (2). V. 177 Providence (R.I.): Amer. Math. Soc., (1996) 159-193. https://doi.org/10.1090/trans2/177.

2. V.A. Malishev and R.A. Minlos, Linear infinite-particle operators // Translations of Mathematical Monographs, 143 (American Mathematical Society, Providence, RI) (1995) 298.

3. D.C. Mattis, The few-body problem on a lattice // Rev. Modern Phys. 58, (1986) 361–379. https://doi.org/10.1103/RevModPhys.58.361.

4. A. Mogilner, Hamiltonians in solid state physics as multi-particle discrete Schrödinger operators: Problems and results // Advances in Soviet Mathematics 5, (1991) 139–194.

5. K.O. Friedrichs, Perturbation of spectra in Hilbert space // AMS., Providence, Rhode Island (1965) 178.

6. M. Moreghebi, Ch. Buhler, S. Yunoki, and A. Moreo, Fermi Surface and Spectral Functions of a Hole Doped Spin-Fermion Model for Cuprates // Phys. Rev. Lett. 84 (12) (2000) 2690–2693. https://link.aps.org/doi/10.1103/PhysRevB.63.214513.

7. I.M. Sigal, A. Soffer, L. Zielinski, On the spectral properties of Hamiltonians without conservation of the particle number // J. Math. Phys. 43 (4), (2002) 1844–1855. https://doi.org/10.1063/1.1452302.

8. Yu. Zhukov, R. Minlos, Spectrum and scattering in a "spin-boson" model with not more than three photons. Theoret. and Math. Phys. 103 (1), (1995) 398–411. https://doi.org/10.1007/BF02069784.

9. S. Albeverio, S.N. Lakaev, T.H. Rasulov, On the Spectrum of a Hamiltonian in Fock Space. Discrete Spectrum asymptotics // Journal of Statistical Physics, 127 (2), (2007) 191–220. https://doi.org/10.1007/s10955-006-9240-6.

10. S. Albeverio, S.N. Lakaev, T.H. Rasulov, The Efimov Effect for a Model Operator Associated to a System of three non-Conserved Number of Particles // Methods of Functional Analysis and Topology, 13 (1), (2007) 1–16. https://mathscinet.ams.org/mathscinet/relay-station?mr=MR2308575.

11. S. Albeverio, S.N. Lakaev, and Z.I. Muminov, Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics // Ann. Henri Poincaré 5, (2004) 743–772. https://doi.org/10.1007/s00023-004-0181-9.

12. M. Reed, B. Simon, Methods of Modern Mathematical Physics. Vol. IV: Analysis of Operators // Academic Press, New York (1978).