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SPECTRAL AND THRESHOLD ANALYSIS OF THE LAPLACIAN WITH NON-LOCAL POTENTIAL IN FOUR-DIMENSIONAL LATTICE

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Abstract: *Eigenvalue behaviour of a family of discrete Schrödinger operators H_α depending on parameter $\alpha \in \mathbb{R}$ is studied on the three-dimensional lattice $\mathbb{Z}^4$. The non-local potential is described by the Kronecker delta function and the shift operator. The characteristics of the Fredholm determinant at values of z below the essential spectrum and their dependence on the parameters are studied. We also show that the essential spectrum absorbs the threshold eigenvalue and threshold resonance.*

Key words: *Discrete Schrödinger operators, essential spectrum, threshold resonance, eigenvalues, lattice.*

I. Introduction

Cladifying Schrödinger operators' spectral properties is one of the most fierce research areas within mathematical physics and operator theory (for recent results see [1-7]). It gives us to better understand the physical processes integrated to those operators. Especially, In lattices Schrödinger operators' eigenvalue behaviors were discussed in many works [8-13], provided the potential is the Dirac delta function.

In this paper we set a goal to explore the spectrum of the discrete Schrödinger operator with a non-local potential given at the points $x_0, -x_0 \in \mathbb{Z}^4$. We demonstrate the feature of Fredholm determinant outside the essential spectrum depending on the parameter α, and the sum of coordinates of the point $x_0 \in \mathbb{Z}^4$. We show clearly by Theorem 1 that the existence conditions for the point $z = 0$ to be an eigenvalue or resonance for a given operator and their parameter α depend on them.

The case of Schrödinger operator given by the non-local potential at one point $x_0 \in \mathbb{Z}^3$ and parameters λ and μ was studied in work [14]. In [15] thorough image of the discrete spectrum of identical operators was described on $\mathbb{Z}^d$ for all dimensions $d \geq 1$. When reciprocity is present at both points x_0 and $-x_0$, the results similar to the cases described above, but adding of α makes the problem more challenging and expands its application potential. In [17] the case of Laplacian operator given by the non-local potential at two points $\pm x_0 \in \mathbb{Z}^3$ and parameter λ was studied. In [18] and [19] the case of Schrödinger operator given by the non-local potential at two points $\pm x_0 \in \mathbb{Z}^3, \pm x_0 \in \mathbb{Z}^d$ and parameters λ and μ was studied. Also, condition for the existence of eigenvalues outside the essential spectrum and on the threshold of the essential spectrum were found.

II. The discrete Schrödinger operator

II.I. Position representation of the discrete Schrödinger operator

For brevity, we use the following notations throughout the paper: $\mathbb{Z}^4$ is the 4–dimensional lattice and $\mathbb{T}^4=(\mathbb{R}/2\pi\mathbb{Z})^4=(-\pi,\pi]^4$ is the 4-dimensional torus (the first Brillouin zone, i.e., the dual group of $\mathbb{Z}^4$) equipped with the Haar measure.

Let $T(y), y\in\mathbb{Z}^4$ be the shift operator

$$(T(y)f)(x)=f(x+y),\quad f\in\ell_2(\mathbb{Z}^4), x\in\mathbb{Z}^4.$$

the discrete Laplacian $\widehat{\Delta}$ on the lattice $\mathbb{Z}^4$ is described by the self-adjoint (bounded) multidimensional Toeplitz-type operator on the Hilbert space $\ell_2(\mathbb{Z}^4)$ ([16]) as

$$\widehat{\Delta}=\frac{1}{2}\sum_{\substack{s\in\mathbb{Z}^4\\|s|=1}}(T(s)-T(0)).$$

Let V_0 be a multiplication operator in $\ell_2(\mathbb{Z}^4)$ by the Kronecker delta function $\delta[\cdot,0]$:

$$V_0f(x)=\delta[x,0]f(x).$$

Then, for a given point $x_0\in\mathbb{Z}^4$, we define the non-local potential (see [16]) as

$$\widehat{V}_{x_0}=\frac{\alpha}{2}(V_0T(x_0)+T^*(x_0)V_0)+\frac{\alpha}{2}(V_0T(-x_0)+T^*(-x_0)V_0).$$

The discrete Schrödinger operator $\widehat{H}_\alpha$ acting in $\ell_2(\mathbb{Z}^4)$, in the position representation, is defined as a bounded self-adjoint perturbation of $-\widehat{\Delta}$ and is of the form

$$\widehat{H}_\alpha=-\widehat{\Delta}-\widehat{V}_{x_0}.$$

II.II. Momentum representation of the discrete Schrödinger operator

In the momentum representation, the one-particle Hamiltonian H_α can be expressed as

$$H_\alpha=H_0-V_{x_0},$$

where H_0 and V_{x_0} are respectively defined as

$$H_0=\mathcal{F}^*(-\widehat{\Delta})\mathcal{F}\quad\text{and}\quad V_{x_0}=\mathcal{F}^*(\widehat{V}_{x_0})\mathcal{F},$$

with $\mathcal{F}$ being the standard Fourier transform $\mathcal{F}:L_2(\mathbb{T}^4)\to\ell_2(\mathbb{Z}^4)$ and $\mathcal{F}^*:\ell_2(\mathbb{Z}^4)\to L_2(\mathbb{T}^4)$ being its inverse. Explicitly, the non-perturbed operator H_0 acts on $L_2(\mathbb{T}^4)$ as a multiplication operator by the function $\mathfrak{e}(\cdot)$:

$$(H_0f)(p)=\mathfrak{e}(p)f(p),\quad f\in L_2(\mathbb{T}^4),$$

where $\mathfrak{e}(p)=\sum_{j=1}^4(1-\cos p_j), p\in\mathbb{T}^4$. The function $\mathfrak{e}(\cdot)$, being a real valued-function on $\mathbb{T}^4$, is referred as the *dispersion relation* of the Laplace operator in the physical literature.

The perturbation V_{x_0} acts on $f\in L_2(\mathbb{T}^4)$ as the two-dimensional integral operator:

$$(V_{x_0}f)(p)=\frac{1}{(2\pi)^4}\int_{\mathbb{T}^4}\frac{\alpha}{2}\left(e^{\mathrm{i}(x_0,p)}+e^{\mathrm{i}(-x_0,p)}+e^{\mathrm{i}(-x_0,s)}+e^{\mathrm{i}(x_0,s)}\right)f(s)ds,$$

which can be rewritten in a more convenient way as

$$(V_{x_0}f)(p)=\frac{1}{(2\pi)^4}\int_{\mathbb{T}^4}\alpha(\cos(x_0,p)+\cos(x_0,s))f(s)ds,\quad f\in L_2(\mathbb{T}^4).$$

II.III. The essential spectrum of H_α

The perturbation V_{x_0} of H_0 is a two dimensional operator, therefore in accordance with the Weyl theorem on the stability of the essential spectrum, the equality $\sigma_{\text{ess}}(H_\alpha) = \sigma(H_0) = \sigma_{\text{ess}}(H_0)$ holds. As H_0 is the multiplication operator by the continuous function $\mathfrak{e}(\cdot)$,

$$\sigma_{\text{ess}}(H_\alpha) = [\mathfrak{e}_{\min}, \mathfrak{e}_{\max}] = [0,8].$$

II.IV. The Fredholm determinant associated with H_α

First, for a complex number $z \in \mathbb{C} \setminus [0,8]$, let us introduce the following integrals

$$A(z) = \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \frac{1}{\mathfrak{e}(t) - z} dt,$$

$$B(z) = \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \frac{\cos(x_0, t)}{\mathfrak{e}(t) - z} dt,$$

$$C(z) = \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \frac{\cos^2(x_0, t)}{\mathfrak{e}(t) - z} dt.$$

Then, for any $\alpha \in \mathbb{R}$, the Fredholm determinant associated to the operator H_α is defined as a regular function in $z \in \mathbb{C} \setminus [\mathfrak{e}_{\min}, \mathfrak{e}_{\max}]$:

$$\Delta(\alpha; z) = \frac{1}{D(z)} - 2\alpha \frac{B(z)}{D(z)} + \alpha^2, \quad D(z) = B^2(z) - A(z)C(z). \quad (1)$$

Lemma 1. *The number $z \in \mathbb{C} \setminus [0,8]$ is an eigenvalue of H_α if and only if $\Delta(\alpha; z) = 0$.*

Proof. Consider the eigenvalue equation

$$(H_\alpha - zI)f = 0,$$

which can be rewritten in a more explicit form as

$$[\mathfrak{e}(p) - z]f(p) - \frac{\alpha \cos(x_0, p)}{(2\pi)^4} \int_{\mathbb{T}^4} f(s)ds - \frac{\alpha}{(2\pi)^4} \int_{\mathbb{T}^4} \cos(x_0, s) f(s) ds = 0,$$

with $f \in L_2(\mathbb{T}^4)$. Denote

$$C_1 = \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} f(t)dt, \quad C_2 = \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \cos(x_0, t) f(t) dt$$

Then, the above equation is equivalent to the system of linear equations with respect to C_1 and C_2

$$\begin{cases} (1 - \alpha B(z))C_1 - \alpha A(z) C_2 = 0 \\ -\alpha C(z) C_1 + (1 - \alpha B(z)) C_2 = 0. \end{cases} \quad (2)$$

The solution f and the solution C_1, C_2 of (2) are related as

$$f(p) = \frac{1}{\mathfrak{e}(p) - z} (\alpha \cos(x_0, p))(C_1 + \alpha C_2).$$

The Fredholm determinant of the system of linear equations (2) is of the form (1), and this fact completes the proof.

III. Properties of $\Delta(\alpha; z)$

For a fixed $x \in \mathbb{Z}^4$ we consider the function

$$R(x, z) = \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \frac{e^{i(x,t)}}{\mathfrak{e}(t) - z} dt, \quad z \in (-\infty, \mathfrak{e}_{\min}), \quad (3)$$

then the functions $A(z), B(z)$ and $C(z)$ can be expressed as $A(z) = R(0,z)$, $B(z) = R(x_0, z)$ and $C(z) = \frac{1}{2}(R(0,z) + R(x_0,z))$ respectively. For the readers convenience we state the lemma from [14] which reveals some useful properties of the function $R(x,z)$:

Lemma 2. *For any $x \in \mathbb{Z}^4$, $R(x,z)$, as a function of z, is positive and monotonically increasing in $(-\infty, 0)$. Moreover, the following asymptotical relation holds*

$$R(x,z) = O\left(\frac{1}{|z|^{|x|_1+1}}\right) \quad as \quad z \to -\infty,$$

where

$$|x|_1 = |x_1| + |x_2| + |x_3| + |x_4|$$

and

$$\lim_{z \to \mathfrak{e}_{\min}} R(x,z) = R(x, \mathfrak{e}_{\min}).$$

The analogy of proof of this lemma can be found [17].

Lemma 3. *The functions $A(z), B(z), C(z)$ and $D(z)$ are monotonically increasing and positive in $(-\infty, 0)$, and the followings are valid*

$$\lim_{z \to 0-} A(z) = A(\mathfrak{e}_{\min}),$$

$$\lim_{z \to 0-} B(z) = B(\mathfrak{e}_{\min}),$$

$$\lim_{z \to 0-} C(z) = C(\mathfrak{e}_{\min}),$$

$$\lim_{z \to 0-} D(z) = D(\mathfrak{e}_{\min}).$$

We also have the aymptotic relations

$$A(z) = O\left(\frac{1}{|z|}\right) \quad as \quad z \to -\infty,$$

$$B(z) = O\left(\frac{1}{|z|^{|x_0|_1+1}}\right) \quad as \quad z \to -\infty,$$

$$C(z) = O\left(\frac{1}{|z|}\right) \quad as \quad z \to -\infty,$$

$$D(z) = O\left(\frac{1}{|z|^2}\right) \quad as \quad z \to -\infty,$$

$$\frac{A(z)}{D(z)} = O(|z|) \quad as \quad z \to -\infty. \tag{4}$$

Proof. Proofs of the statements involving the functions $A(z)$ and $B(z)$ follow from the equalities $A(z) = R(0,z)$, $B(z) = R(x_0, z)$ and Lemma 2. The relation

$$D(z) = B^2(z) - A(z)C(z),$$

the limits

$$A_0 = \lim_{z \to 0-} A(z) = \frac{1}{(2\pi)^4} \int_{\mathbb{T}^4} \frac{1}{\mathfrak{e}(t) - \mathfrak{e}_{\min}} dt,$$

$$B_0 = \lim_{z\to 0-} B(z) = \frac{1}{(2\pi)^4}\int_{\mathbb{T}^4} \frac{\cos(x_0,t)}{\mathfrak{e}(t)-\mathfrak{e}_{\min}}dt,$$

$$C_0 = \lim_{z\to 0-} C(z) = \frac{1}{(2\pi)^4}\int_{\mathbb{T}^4} \frac{\cos^2(x_0,t)}{\mathfrak{e}(t)-\mathfrak{e}_{\min}}dt$$

and the properties of $A(z)$ and $B(z)$ yield the proof of the statements related to $D(z)$.

Lemma 4. *(a) For any $z \in (-\infty, \mathfrak{e}_{min})$, the numbers*

$$\alpha_2(z) = \frac{1}{B(z)+\sqrt{A(z)C(z)}} \quad and \quad \alpha_1(z) = \frac{1}{B(z)-\sqrt{A(z)C(z)}} \qquad (5)$$

are α-intercepts.

(b) For any $\xi, z \in (-\infty, \mathfrak{e}_{\min})$ with $\xi < z$, the inequalities

$$\alpha_1(\xi) < \alpha_1(z) < 0 < \alpha_2(z) < \alpha_2(\xi) \qquad (6)$$

and

$$|\alpha_1(z)| > \alpha_2(z) \qquad (7)$$

hold.

Moreover, we have

$$\alpha_1^0 := \lim_{z\to\mathfrak{e}_{\min}-} \alpha_1(z) = \frac{1}{B_0-\sqrt{A_0C_0}} < 0, \quad \lim_{z\to-\infty} \alpha_1(z) = -\infty \qquad (8)$$

and

$$\alpha_2^0 := \lim_{z\to\mathfrak{e}_{\min}-} \alpha_2(z) = \frac{1}{B_0+\sqrt{A_0C_0}}, \quad \lim_{z\to-\infty} \alpha_2(z) = +\infty. \qquad (9)$$

Proof. Simple calculations yield the statement (a).

(b) Due to Lemma 3, the functions $A(z) \pm B(z)$ are monotonically increasing in the interval $(-\infty, \mathfrak{e}_{\min})$, therefore the relations

$$\sqrt{A(z)C(z)} + B(z) > \sqrt{A(\xi)C(\xi)} + B(\xi) > 0 > B(\xi) - \sqrt{A(\xi)C(\xi)} > B(z) - \sqrt{A(z)C(z)}$$

and

$$0 > B(z) - \sqrt{A(z)C(z)} > -(\sqrt{A(z)C(z)} + B(z))$$

provide the proof of inequalities (6) and (7).

IV. Threshold eigenvalues and threshold resonances of H_α

So far, we have studied the equation $H_\alpha f = zf$ for $z \in (-\infty, \mathfrak{e}_{\min})$. Now, we consider it at the left edge $z = \mathfrak{e}_{\min}$ of the essential spectrum.

Definition.

In the equation $H_\alpha f = \mathfrak{e}_{min} f$, $\mathfrak{e}_{min}$ is called

- a lower threshold eigenvalue if $f \in L_2(\mathbb{T}^4)$,
- a lower threshold resonance if $f \in L_1(\mathbb{T}^4)\backslash L_2(\mathbb{T}^4)$,
- a lower super-threshold resonance if $f \in L_\epsilon(\mathbb{T}^4)\backslash L_1(\mathbb{T}^4)$ for any $0 < \epsilon < 1$.

If $H_\alpha f = \mathfrak{e}_{\min} f$ has no solutions in $L_1(\mathbb{T}^4)$, then $\mathfrak{e}_{\min}$ is a regular point of H_α.

For a continuous function

$$\varphi(p) = C_3\cos(x_0,p) + C_4$$

(where $C_3, C_4 \in \mathbb{C}$ are fixed numbers) define

$$g(p) = \varphi(p)/\mathfrak{e}(p).$$

The function $1/\mathfrak{e}(p)$ has a unique singular point at the origin $p = 0$, and approximated as $\mathfrak{e}(p) \approx |p|^2$ at this point. The lemma below is a straightforward consequence of the definition of g and the properties of $\mathfrak{e}(\cdot)$

Lemma 5.

- if $\varphi(0) = 0$, then $g \in L_2(\mathbb{T}^4)$.
- if $\varphi(0) \neq 0$, then $g \in L_1(\mathbb{T}^4) \backslash L_2(\mathbb{T}^4)$.

Proof. From the Makloren series expansion of the function $y = \cos x$, the equation

$$\varphi(p) = C_3 + C_4 + (x_0, p)^2 \psi(p) \tag{10}$$

can be written, where $\psi(p)$ is a continuous function. At the same time, it is possible to show

$$\frac{2}{\pi^2} p^2 \leq \mathfrak{e}(p) \leq \frac{1}{2} p^2, \quad p \in \mathbb{T}^4 \tag{11}$$

from the relation $\frac{2x}{\pi} \leq \sin x \leq x, \quad 0 \leq x \leq \frac{\pi}{2}$.

(a) If $\varphi(0) = 0$, then such numbers $B_1 > 0$ and $\delta > 0$ are found, $|\varphi(p)| \leq B_1 |p|^2, \quad p \in \mathbb{T}^4$ and

$$\int_{\mathbb{T}^4} (g(p))^2 dp = \int_{\mathbb{T}^4} \frac{(\varphi(p))^2}{\mathfrak{e}(p)^2} dp \leq B_1 \int_{T^4} \frac{p^4}{(\frac{2}{\pi^2} p^2)^2} dp = B_2 \int_{\mathbb{T}^4} 1 dp < \infty.$$

It means that $g \in L_2(\mathbb{T}^4)$.

(b) Let $\varphi(0) \neq 0$, $B_1 > 0$ and $\delta > 0$ numbers. There exist number $B_1 > 0$ and $\delta > 0$, such that

$$|\varphi(p)| \geq B_1, \quad p \in U_\delta(0).$$

Then from (11)

$$\int_{\mathbb{T}^4} (g(p))^2 dp \geq \int_{U_\delta(0)} (g(p))^2 dp \geq \frac{B_1^2}{\frac{1}{4}} \int_{U_\delta(0)} \frac{dp}{p^4} = \infty,$$

and so $g \notin L_2(\mathbb{T}^4)$.

The inequality (11) allows to get estimation.

$$\int_{\mathbb{T}^4} |g(p)| dp \leq \int_{\mathbb{T}^4} \frac{M_0}{\frac{2}{\pi^2 p^2}} dp < \infty.$$

where $M_0 \geq \max|\varphi(p)|$. Namely $g \in L_1(\mathbb{T}^4)$, so $g \in L_1(\mathbb{T}^4) \backslash L_2(\mathbb{T}^4)$.

In the theorem below, we describe the conditions for $\mathfrak{e}_{\min} = 0$ to be a regular point, an eigenvalue or a threshold resonance.

Theorem. *(a) Let $\Delta(\alpha, 0) \neq 0$.*

The number 0 is a regular point of H_α.

(b) Let $\Delta(\alpha, 0) = 0$.

- The number 0 is an embedded eigenvalue of H_α, if $\alpha = \frac{1}{A_0 + B_0}$;
- The number 0 is a threshold resonance of H_α, if $\alpha \neq \frac{1}{A_0 + B_0}$.

Proof.

(a) For the equation

$$H_\alpha f = 0 \tag{12}$$

has a solution, if and only if the system of equation

$$\begin{cases}(1-\alpha B_0)C_5 - \alpha A_0 C_6 = 0\\ -\alpha C_0 C_5 + (1-\alpha B_0)C_6 = 0.\end{cases} \quad (C_5, C_6) \in \mathbb{C}^2 \tag{13}$$

has a solution, where the solutions of (12) and (13) are connected by relations

$$f(p) = \frac{\alpha}{\epsilon(p)}\varphi(p), \quad \varphi(p) = \cos(x_0, p)C_5 + C_6, \tag{14}$$

and

$$C_5 = \frac{1}{(2\pi)^4}\int_{\mathbb{T}^4} f(s)ds, \quad C_6 = \frac{1}{(2\pi)^4}\int_{\mathbb{T}^4} \cos(x_0, s)f(s)ds.$$

Since the determinant $\Delta(\alpha, 0)$ of the system (13) is not zero, equation (12) has nontrivial solution, that is, the number 0 is a regular point of the operator H_α.

[(b)] Since $\Delta(\alpha, 0) = 0$ system of equations (13) has a unique solution, and according to (14)

$$f(p) = \frac{\alpha}{\epsilon(p)}\varphi(p), \quad \varphi(p) = \cos(x_0, p)C_5 + C_6$$

where $f(p)$ is a solution of (12).

- (b1) $\alpha = \frac{1}{A_0 + B_0}$ if and only if $\varphi(0) = 0$ it follows from Lemma 5 that $f \in L_2(\mathbb{T}^4)$.

- (b2) $\alpha \neq \frac{1}{A_0 + B_0}$ if and only if $\varphi(0) \neq 0$ it follows from Lemma 5 that $f \in L_1(\mathbb{T}^4) \backslash L_2(\mathbb{T}^4)$.

The last relation and the definition complete the proof.

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