DOI: 10.5281/zenodo.15033820
Link: https://zenodo.org/records/15033820

ASYMPTOTIC BEHAVIOR OF EIGENVALUES IN DISCRETE SCHRÖDINGER OPERATORS WITH POINT INTERACTION POTENTIAL

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Abstract. *We investigate the one-particle discrete Schrödinger operator with Dirac delta potential on the d-dimensional cubic lattice. We show that the operator the operator has a unique eigenvalue and obtain an asymptotics expansion for this eigenvalue as wight of potential approaches the infinity.*
Keywords: *Schrödinger operator, spectrum, eigenvalue, Fredholm determinant, eigenvalue asymptotics.*

I.INTRODUCTION

The spectral properties of discrete Schrödinger operators have attracted significant attention in the study of both combinatorial Laplacians and quantum graphs (see [3], [4], [6], [9], [11], [15], [21], and references therein for recent summaries). In particular, the eigenvalue behavior of Schrödinger operators on lattices has been investigated in [1], [5], [8], [12], [13], and briefly discussed in [10] and [16], under the assumption that the potential is a Dirac delta function.

In addition, the spectral properties of the Hamiltonian describing the motion of a single quantum particle on a lattice in an external field were studied in [12, 16, 17, 19, 20]. These works investigated the number of eigenvalues and their locations, depending on the value of the interaction energies.

In the present paper, we study spectral properties of the discrete Schrödinger operator $\widehat{H}_{\mu}$, $\mu \in \mathbb{R}$, on the d- dimensional lattice with a zero-range potential (Dirac delta potential) concentrated around $x = 0 \in \mathbb{Z}^{\mathrm{d}}$ and with an extended dispersion relation depending on the real parameter β ($\beta > 0$).

In the case when the potential is δ function, the existence and uniqueness of $z(\mu)$ have been investigated in the [1, 13] and the Hamiltonian of a system of three quantum mechanical particles moving on the three-dimensional lattice $\mathbb{Z}^3$ and interacting via zero-range attractive potentials is considered in the [2, 16, 19]. In [14], asymptotical behaviour of the eigenvalues as $\mu \to 0$ was studied.

Note that the authors of [24] have investigated this operator, and they proved that the operator $\widehat{H}_\mu$ has a unique eigenvalue $z(\mu,\beta)$ for each value of the parameter μ. In addition, they showed that $z(\mu,\beta)$ is infinitely many times differentiable as the function of μ for $d = 1,2$, and established the asymptotics for this eigenvalue as $\mu \to \infty$.

In the present paper, we improve the results obtained in [24], that is, we show that $z(\mu,\beta)$ is infinitely many times differentiable as the function of μ for any dimension of the lattice $\mathbb{Z}^d$ and we get the more accurate asymptotics of eigenvalues as $\mu \to \infty$

Note that, to the best of our knowledge, this result is new in this field.

In order to facilitate the description of the content, we introduce the followius notations used throughout this manuscript: $\mathbb{Z}^d$ is the d–dimensional lattice and $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d = (-\pi,\pi]^d$ is the d-dimensional torus equipped with the Haar measure.

II.Discrete Schrödinger operator

II.I. Discrete Schrödinger operator in the position representation.

For each $\beta > 0$, Laplacian Δ is defined as follows:

$$\Delta = \frac{1}{2}\sum_{j=1}^{d} \left((\hat{T}(j) + \hat{T}^*(j)) - 2\hat{T}(0) + \beta(\hat{T}(2j) + \hat{T}^*(2j) - 2\hat{T}(0))\right),$$

where $\hat{T}(y)$ is a shift operator on $\ell^2(\mathbb{Z}^d)$ by $y \in \mathbb{Z}^d$:

$\hat{T}(y)\hat{f}(x) = \hat{f}(x+y), \quad \hat{f} \in \ell^2(\mathbb{Z}^d).$

Let us define the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ was

$$\widehat{H}_\mu = -\Delta - \hat{V}_\mu,$$

where the potential $\hat{V}$ depends on the parameter $\mu \in \mathbb{R}$ and satisfies the relation

$$(\hat{V}_\mu\hat{f})(x) = \begin{cases} \mu\hat{f}(x), & if \quad x = 0 \\ 0 & if \quad x \neq 0' \end{cases} \quad \hat{f} \in \ell^2(\mathbb{Z}^d), \quad x \in \mathbb{Z}^d$$

II.II. Discrete Schrödinger operator in the momentum representation.

The one-particle Schrödinger operator H_μ in the momentum representation is defined as

$$H_\mu = \mathcal{F}^*\widehat{H}_\mu\mathcal{F}, \quad H_\mu = H_0 - V_\mu,$$

where

$$H_0 = \mathcal{F}^*(-\triangle)\mathcal{F}, \quad V_\mu = \mathcal{F}^*\hat{V}_\mu\mathcal{F}. \qquad (1)$$

Here, $\mathcal{F}$ stands for the standard Fourier transformation $\mathcal{F}: L^2(\mathbb{T}^d) \to \ell^2(\mathbb{Z}^d)$ with the inverse $\mathcal{F}^*: \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$. Explicitly, the non-perturbed operator H_0 acts on $L^2(\mathbb{T}^d)$ as a multiplication operator by the function $\varepsilon(\cdot)$:

$$(H_0 f)(p) = \varepsilon(p)f(p), \quad f \in L^2(\mathbb{T}^d), \quad p \in \mathbb{T}^d,$$

where

$$\varepsilon(p) = \sum_{j=1}^{d} (1 - \cos p_j) + \beta(1 - \cos 2p_j), \quad p \in \mathbb{T}^d, \quad \beta \geq 0. \qquad (2)$$

In the physical literature, the function $\varepsilon(\cdot)$, being a real valued-function on $\mathbb{T}^d$, is called the dispersion relation of the Laplacian operator $-\triangle$. ()

The potensial operator is transformed into a rank one integral operator

$$(Vf)(p) = \frac{\mu}{(2\pi)^d} \int_{\mathbb{T}^d} f(q)dq, \quad f \in L^2(\mathbb{T}^d). \qquad (3)$$

Now, we present the lemma on the range of the function $\varepsilon(p)$ defined in the form (2.3).

Lemma 1. *a) For any* $\boldsymbol{\beta > 0}$*, the function* $\boldsymbol{\varepsilon(p)}$ *has a non-oscillating minimum at the point* $\boldsymbol{Q_0 = (0, 0, \ldots, 0)}$ *and* $\boldsymbol{\varepsilon(Q_0) = 0}$.

b1) If $0 < \beta < \frac{1}{4}$, *then the function* $\varepsilon(p)$ *has* a non-oscillating maximum at the point $Q_1 = (\pi, \pi, \ldots, \pi)$ *and* $\varepsilon(Q_1) = 2d$.

b2) When $\boldsymbol{\beta = \frac{1}{4}}$, ***the function*** $\boldsymbol{\varepsilon(p)}$ ***has*** **oscillating maximum at the point** $\boldsymbol{Q_1 = (\pi, \pi, \ldots, \pi)}$ ***and*** $\boldsymbol{\varepsilon(Q_1) = 2d}$.

b3) When $\beta > \frac{1}{4}$, *the fuction* $\varepsilon(p)$

$Q_2 = \left(arccos\left(-\frac{1}{4\beta}\right), arccos\left(-\frac{1}{4\beta}\right), \ldots, arccos\left(-\frac{1}{4\beta}\right)\right)$ *has* a non-oscillating maximum at the point $\varepsilon(Q_2) = \sum_{i=1}^{d} \frac{(1+4\beta)^2}{8\beta} = \frac{(1+4\beta)^2 d}{8\beta}$.

Proof. a) The critical points of the function $\varepsilon(p)$, i.e., the solutions of the equation

$$\frac{\partial \varepsilon(q)}{\partial q_i} = \sin q_i (1 + 4\beta \cos q_i) = 0$$

consist of $Q_0 = (0,0, \ldots, 0)$ and $Q_1 = (\pi, \pi, \ldots, \pi)$, when $\beta \neq \frac{1}{4}$, and

$$Q_2 = \left(arccos\left(-\frac{1}{4\beta}\right), arccos\left(-\frac{1}{4\beta}\right), \ldots, arccos\left(-\frac{1}{4\beta}\right)\right)$$

when $\beta > \frac{1}{4}$. In this case, the Hessian matrix

$$H_\varepsilon(Q_0) = \begin{pmatrix} 1 + 4\beta & 0 & \ldots & 0 & 0 \\ 0 & 1 + 4\beta & & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & 1 + 4\beta \end{pmatrix}$$

which consists of second-order partial derivatives

$$\frac{\partial^2 \varepsilon(q)}{\partial q_i \partial q_j} = 0, i \neq j, i,j = \overrightarrow{1,d},$$

$$\frac{\partial^2 \varepsilon(q)}{\partial^2 q_i} = \cos q_i + 4\beta \cos 2q_i$$

is positive definite at the point Q_0. Therefore, the function ε(q) has a global minimum at the point Q_0, and we can see that $\varepsilon(Q_0) = 0$.

b1) Let $0 < \beta < \frac{1}{4}$. Then the Hessian matrix at the point Q_1

$$H_\varepsilon(Q_1) = \begin{pmatrix} -1+4\beta & 0 & \dots & 0 & 0 \\ 0 & -1+4\beta & & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & -1+4\beta \end{pmatrix}$$

Is negative definite, therefore, in this case the function ε(q) has global maximum at the point Q_1 and $\varepsilon(Q_1) = 2d$.

b2) When $\beta = \frac{1}{4}$, Hessian matrix $H_\varepsilon(Q_1) = 0$, so the number $\varepsilon(Q_1) = 2d$ is a local maximum of the function ε(q) .

b3) When $\beta > \frac{1}{4}$, Hessian matrix at the point Q_2

$$H_\varepsilon(Q_2) = \begin{pmatrix} \frac{1-16\beta^2}{4\beta} & 0 & \dots & 0 & 0 \\ 0 & \frac{1-16\beta^2}{4\beta} & & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \frac{1-16\beta^2}{4\beta} \end{pmatrix}$$

is negative defnite and
$\varepsilon(Q_1) < \varepsilon(Q_2)$.

That is, when $\beta > \frac{1}{4}$, the number $\varepsilon(Q_2) = \sum_{i=1}^{d} \frac{(1+4\beta)^2}{8\beta} = \frac{(1+4\beta)^2 d}{8\beta}$ is a global maximum value of the function ε(p). Therefore the minimum and maximum of ε(p) consists of

$$\varepsilon_{\min} = 0 \quad \text{and } \varepsilon_{\max} = \begin{cases} 2d, & 0 \leq \beta \leq \frac{1}{4}, \\ \frac{(1+4\beta)^2 d}{8\beta}, & \beta > \frac{1}{4}. \end{cases}$$

III. The essential spectrum.

H_0 is a multiplication operator by the real valued, continuous function and the perturbation V is the one-dimensional operator and, therefore, in accordance to the Weyl theorem on the stability of the essential spectrum the equality $\sigma_{ess}(H_\mu) = \sigma_{ess}(H_0)$ holds the essential spectrum of the operator H_μ consists of the following segment on the real axis:

$$\sigma_{ess}(H_\mu) = [\varepsilon_{\min}, \varepsilon_{\max}],$$

IV. Fredholm determinant of the operator H_μ

For any $\mu \in \mathbb{R}$, we define the Fredholm determinant associated with the operator H_μ as a regular function in $z \in \mathbb{R}\backslash[\varepsilon_{\min}, \varepsilon_{\max}]$ as

$$D(\mu,z) = 1 - \frac{\mu}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{dp}{\varepsilon(p)-z}. \qquad (4)$$

Lemma 2. *The number $z \in \mathbb{C}\backslash[\varepsilon_{min}, \varepsilon_{max}]$ is an eigenvalue of H_μ if and only if $D(\mu,z) = 0$ and their multiplicities are the same.*

Proof. The eigenvalue equation $(H_\mu - z)f = 0$ i.e., the equation

$$[\varepsilon(p)-z]f(p) - \frac{\mu}{(2\pi)^d}\int_{\mathbb{T}^d} f(p)dp = 0 \qquad (5)$$

for $f \in L^2(\mathbb{T}^d)$ is equivalent to the equation

$$\left(1 - \frac{\mu}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{dq}{\varepsilon(q)-z}\right)C_f = 0, \qquad (6)$$

for the unknown coefficient $C_f \in \mathbb{C}$ and solutions of (5) and (6) are related as

$$C_f = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(p)dp$$

and

$$f(p) = \frac{\mu}{\varepsilon(p)-z}C_f.$$

The equation (6) has a non-trivial solution if and only if $D(\mu,z) = 0$.

Lemma 3. a) Let $d = 1,2$. Then

$$I = \lim_{z\to\varepsilon_{\min}-}\int_{\mathbb{T}^d}\frac{dq}{\varepsilon(q)-z} = \int_{\mathbb{T}^d}\frac{dq}{\varepsilon(q)-\varepsilon_{\min}} = +\infty.$$

b) Let $d \geq 3$. In that case the integral

$$I = \int_{\mathbb{T}^d}\frac{dq}{\varepsilon(q)-\varepsilon_{\min}}$$

exists.

Proof. a) Let $d = 1$. Then we have

$$\int_{\mathbb{T}}\frac{dq}{\varepsilon(q)-z} = \int_{\mathbb{T}}\frac{dq}{2\sin^2\frac{q}{2} + 2\beta\sin^2 q - z}$$

$$\geq \int_{\mathbb{T}}\frac{dq}{\frac{q^2}{2} + 2\beta q^2 - z} = \int_{\mathbb{T}}\frac{dq}{q^2 + \frac{2|z|}{1+4\beta}}$$

$$= \frac{1}{\sqrt{\frac{2|z|}{1+4\beta}}}\arctan\frac{q}{\sqrt{\frac{2|z|}{1+4\beta}}} \longrightarrow \infty, \qquad z \longrightarrow 0^-$$

Let $d = 2$. Then we have

$$\int_{\mathbb{T}^2}\frac{dq}{\varepsilon(q)-z} = \int_{\mathbb{T}^2\backslash U_\delta(0)}\frac{dq}{\varepsilon(q)-z} + \int_{U_\delta(0)}\frac{dq}{\varepsilon(q)-z};$$

$$\int_{\mathbb{T}^2\setminus U_\delta(0)} \frac{dq}{\varepsilon(q)-z} < \infty,$$

$$\int_{U_\delta(0)} \frac{dq}{\varepsilon(q)-z} = \int_{U_\delta(0)} \frac{dq}{\sum_{j=1}^{2} (1-\cos q_j + \beta(1-\cos 2q_j)) - z} = \int_{U_\delta(0)} \frac{dq}{\frac{1+4\beta}{2} q^2 - z}$$

$$= \int_0^{2\pi}\int_0^{\delta} \frac{r dr d\theta}{\frac{1+4\beta}{2} r^2 - z} = \frac{1}{1+4\beta}\int_0^{2\pi} \ln\left(r^2 - \frac{2z}{1+4\beta}\right)|_0^r d\theta = \infty.$$

b) Let $W_\gamma(0)$ be a sphere in $\mathbb{R}^d$ with center at $y=0$ and radius $\gamma = \frac{1+4\beta}{2\sqrt{2\beta}}$. Consider the mapping $q=\phi(y)$:

$$\phi: W_\gamma(0) \to U(0) \subset \mathbb{T}^d, \quad q_j: \to \arccos \frac{\sqrt{(1+4\beta)^2 - 8\beta y_j^2} - 1}{4\beta}; \; j = 1, \dots, d.$$

where $U(0) := \phi(W_\gamma(0))$ is image of the sphere $W_\gamma(0)$.

It is easy to see that $q=0$ is a unique non-degenerate minimum of $\varepsilon(q)$. We rewrite function I as

$$I = \int_{U(0)} \frac{dq}{\varepsilon(q)} + \int_{\mathbb{T}^d\setminus U(0)} \frac{dq}{\varepsilon(q)} = I_1 + I_2$$

Observe that $I_2 < \infty$. In the first integral making a change of variables $q=\phi(y)$, we obtain

$$I_2 = \int_{W_\gamma(0)} \frac{J(\varphi(y))dy}{\sum_{j=1}^{d} y_j^2}, \qquad (7)$$

where $J(\varphi(y))$ is jacobian of the substition φ. Passing to the spherical coordinates $y = r\psi$ in integral (7), we obtain :

$$I_2 = \int_0^{\gamma} r^{d-3} dr \int_{\Lambda_{d-1}} J(\varphi(r\psi)) d\psi, \qquad (8)$$

where collection Λ_{d-1} is unity sphere in $\mathbb{R}^d$. Then from $d \geq 3$ and the continuity of the functions J, φ, follows the existence of the integral I_2.

Lemma 4. *a) a) If* $d = 1,2$ *or* $d = 3,4$ *and* $\beta = \frac{1}{4}$*, then*

$$d(\varepsilon_{\max}) = \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - \varepsilon_{\max}} = -\infty$$

b) If $d \geq 3$ *and* $\beta \neq \frac{1}{4}$ *or* $d > 4$ *and* $\beta = \frac{1}{4}$*, then*

$$d(\varepsilon_{max}) = \int_{\mathbb{T}^d} \frac{dq}{\varepsilon(q) - \varepsilon_{\max}}$$

exists.

proof. We divide the proof into two parts: $\beta \neq \frac{1}{4}$ and $\beta = \frac{1}{4}$.

If $\beta \neq \frac{1}{4}$, according to lemma 2 on the maximum value of the dispersion function, there exist $C_1 > 0$, $C_2 > 0$ numbers such that

$$C_1|q - Q^*|^2 \leq |\varepsilon(\mathrm{q}) - \varepsilon_{\max}| \leq C_2|q - Q^*|^2, \; q \in \mathbb{T}^d \qquad (9)$$

satisfies, where Q^* is absolute maximum point of the function $\varepsilon(\mathrm{q})$, i.e., $Q^* = (\pi, \dots, \pi) \in \mathbb{T}^d$, if $0 < \beta < 1/4$;

$Q^* = \left(arccos(-1/4\beta), \dots, arccos(-1/4\beta)\right) \in \mathbb{T}^d$, if $\beta > 1/4$.

In the following expressions, if the exact value of a constant number does not affect the calculation, we denote it by $C_1, C_2, \dots$.

a) a) First, let is consider the case when $\mathrm{d} = 1{,}2$. In this case, from the inequality on the right-hand side of 9),

$$|\mathrm{d}(\varepsilon_{\max})| = \frac{1}{(2\pi)} \int_{\mathbb{T}} \frac{dq}{|\varepsilon_{\max} - \varepsilon(q)|} \geq \frac{1}{(2\pi)C_2} \int_{\mathbb{T}} \frac{dq}{\mathrm{q}^2}$$

$$= \frac{1}{(2\pi)C_2} \int_{\mathbb{T}} \frac{dq}{\mathrm{q}^2} \geq C_3 \int_0^{\pi} \frac{1}{q^2} dq = C_3 \frac{1}{q}\Big|_0^1 = -\infty,$$

where C_3 is some positive constant.

Now, when d =2, we estimate the integral

$|d(\varepsilon_{\max})|$ using the right-hand side of the inequality (9) and then transitioning to polar coordinates as follows:

$$|d(\varepsilon_{\max})| = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{dq}{|\varepsilon(q) - \varepsilon_{max}|} \geq C_3 \int_{\mathbb{T}^2} \frac{dq}{|q - Q^*|^2}$$

$$= C_3 \int_{\mathbb{T}^2} \frac{dq}{|q|^2} > C_4 \int_0^1 \frac{1}{r} dr = \frac{1}{(2\pi)^2 C} ln\, r|_0^1 = \infty.$$

b) If $d \geq 3$, we estimate the integral from above by considering the left-hand side of the inequality (9) and then transitioning to the spherical coordinate system:

$$\int_{\mathbb{T}^d} \frac{dq}{|\varepsilon(q) - \varepsilon_{max}|} \leq C_3 \int_{\mathbb{T}^d} \frac{dq}{|q - Q^*|^2} = C_3 \int_{\mathbb{T}^d} \frac{dq}{|q|^2}$$

$$\leq C_4 + C_5 \int_0^{\delta} \rho^{d-2-1}\, d\rho < \infty,$$

where $\delta > 0$ is enaugh small number.

In the remaining part of our proof, we consider case $\beta = 1/4$. It is known that, according to Lemma 2, the number ε_{max} is an attained maximum, and the expression

$$\varepsilon(p) - \varepsilon_{max} = \sum_{i=1}^{d} (-1 - cos\, p_i) + \sum_{i=1}^{d} \frac{1}{2}(1 - cos^2 p_i) =$$

$$- \sum_{i=1}^{d} (1 - cos\, p_i)\left(1 - \frac{1}{2}(1 - cos\, p_i)\right) = - \sum_{i=1}^{d} \frac{1}{2}(1 + cos\, p_i)^2$$

allows us to write inequalities

$$C_1|p - Q_1|^4 \leq |\varepsilon(p) - \varepsilon(Q_1)| \leq C_2|p - Q_1|^4, \qquad p \in \mathbb{T}^d,$$

where $Q_1 = (\pi, \dots, \pi) \in \mathbb{T}^d$.

From the final inequality, we estimate the integral $|d(\varepsilon_{\max})|$ from above:

$$\frac{1}{C_2}\int_{\mathbb{T}^d}\frac{dq}{|q-Q_1|^4}\le\int_{\mathbb{T}^d}\frac{dq}{|\varepsilon(q)-\varepsilon_{max}|}\le\frac{1}{C_1}\int_{\mathbb{T}^d}\frac{dq}{|q-Q_1|^4}\qquad(10)$$

a) After that (if $d=2$, by transitioning to the polar coordinate, agar $d=3,4$, by transitioning to the spherical coordinate) from using the left-hand side of the last inequality, we estimate the integral $|d(\varepsilon_{\max})|$ above:

$$\int_{\mathbb{T}^d}\frac{dq}{|\varepsilon(q)-\varepsilon_{max}|}\ge C_3\int_0^{\pi}\frac{\rho^{d-1}}{\rho^4}d\rho=C_3\int_0^{\pi}\rho^{d-5}\,d\rho=\infty,\ \ d\le 4$$

b) In this case, using the inequality at the right side of (10) and transitioning to the spherical coordinate, we obtain

$$\int_{\mathbb{T}^d}\frac{dq}{|\varepsilon(q)-\varepsilon_{max}|}\le C_3\int_{\mathbb{T}^d}\frac{dq}{|p-Q_1|^4}\le C_4+C_5\int_0^{\delta}\frac{\rho^{d-1}}{\rho^4}d\rho<\infty,$$

where $\delta>0$ a sufficiently small number. The lemma has been completely proven.

Lemma 5. a) For any $\mu>0$ there exists a unique number $z(\mu,\beta)$ in the interval $(-\infty,0)$ such that $D(\mu,z(\mu,\beta))=0$, and $z(\mu,\beta)$ is differentiable with respect to the μ-variable.

b) For any $\mu<0$ there exists a unique number $z(\mu,\beta)$ in the interval $(\varepsilon_{max},\infty)$ such that $D(\mu,z(\mu,\beta))=0$, and $z(\mu,\beta)$ is differentiable with respect to the μ-variable.

Proof. a) For any fixed $\mu>0$, the function $D(\mu,\cdot)$ is continuous on $(-\infty,0)$ and it has the limits

$$\lim_{z\to-\infty}D(\mu,z)=1$$

and by statement a) of Lemma 2.2 yields

$$\lim_{z\to 0^-}D(\mu,z)=-\infty$$

and hence for any $\mu>0$ there exists a unique $z(\mu,\beta)\in(-\infty,0)$ such that

$$D(\mu,z(\mu,\beta))=0.$$

Since $D(\mu,z)>1$ for $z\in(\varepsilon_{max},\infty)$ the function $D(\mu,\cdot)$ has no any zeros on the interval $(\varepsilon_{max},\infty)$.

As $D(\mu,z)$ is differentiable with respect to (μ,z) and

$$\frac{\partial D}{\partial z}=-\frac{\mu}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{dq}{(\varepsilon(q)-z)^2}\ne 0,$$

due to the implicit function theorem the function $z(\mu,\beta)$ is differentiable on $(-\infty,0)$.

b) The proof of this part is same a).

Theorem 1. Let $d(d\ge 3)$ *and*

$$\mu_0=\left(\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{dq}{\varepsilon(q)-0}\right)^{-1},\qquad \mu^0=\left(\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{dq}{\varepsilon(q)-\varepsilon_{\max}}\right)^{-1}.$$

a) If $\mu > \mu_0$, the operator H_μ has a unique eigenvalue $z(\mu,\beta)$in the interval $(-\infty, 0)$ and has no eigenvalue to the right of essential spectrum.

b) If $\mu < \mu^0$, the operator H_μ has a unique eigenvalue $z(\mu,\beta)$in the interval $(\varepsilon_{max}, \infty)$ and has no eigenvalue to the left of essential spectrum.

c) If $\mu^0 \le \mu \le \mu_0$, the operatorH_μ has no an eigenvalue eighter in the interval $(-\infty, 0)$ or $(\varepsilon_{max}, +\infty)$.

Proof. Lemmas 3 and 5 imply the proof.

Lemma 6. $z(\mu,\beta)$ *satisfies the asymptotics*

$$z(\mu,\beta) = \frac{1}{-\frac{1}{\mu} - d(1+\beta)\frac{1}{\mu^2} - \frac{d((2d-1)\beta^2 + 4d\beta + (2d-1))}{2!}\frac{1}{\mu^3} + O(\frac{1}{\mu^4})},$$

as $\mu \longrightarrow \infty$.

Proof. In the r.h.s of (4), we make the substitutions $\mu \longrightarrow \frac{1}{\lambda}$ and $z \longrightarrow \frac{1}{\zeta}$

$$\widetilde{D}(\lambda,\zeta) = D\left(\frac{1}{\lambda}, \frac{1}{\zeta}\right).$$

Instead of the equation $D(\mu, z) = 0$ *as* $\mu \longrightarrow +\infty$ *and* $z \longrightarrow -\infty$. We investigate

$$\widetilde{D}(\lambda,\zeta) = \lambda - \zeta \int_{\mathbb{T}^d} \frac{dq}{\zeta\varepsilon(q) - 1} = 0.$$

Due to the relations

$$\widetilde{D}(0,0) = 0 \quad and \quad \frac{\partial \widetilde{D}(0,0)}{\partial \zeta} = 1$$

we may apply the implicit function theorem to define the first three coefficients of the Taylor series

$$\zeta = \zeta(\lambda) = \zeta(0) + \zeta'(0)\lambda + \frac{\zeta''(0)}{2!}\lambda^2 + \frac{\zeta'''(0)}{3!}\lambda^3 + \cdots \quad .$$

at the point $\lambda = 0$. Thus the implicit theorem allows to write down the equalities

$$\zeta(0) = 0,$$

$$\zeta'(0) = -1.$$

Similarly,

$$\zeta''(0) = -2d(1+\beta), \quad \zeta'''(0) = -3d((2d-1)\beta^2 + 4d\beta + (2d-1)).$$

Hence, we get

$$\zeta = -\lambda - d(1+\beta)\lambda^2 - \frac{d((2d-1)\beta^2 + 4d\beta + (2d-1))}{2!}\lambda^3 + O(\lambda^4) \quad as \quad \lambda \longrightarrow 0.$$

Making the substitutions $\mu = \frac{1}{\lambda}$ and $z = \frac{1}{\zeta}$, we get the following result:

$$z = \frac{1}{-\frac{1}{\mu} - d(1+\beta)\frac{1}{\mu^2} - \frac{d((2d-1)\beta^2 + 4d\beta + (2d-1))}{2!}\frac{1}{\mu^3} + O(\frac{1}{\mu^4})}.$$

Taking into the account the reciprocal relations $\lambda = \frac{1}{\mu}$ and $\zeta = \frac{1}{z}$ we obtain the following assertion.

Theorem 2. $z(\mu,\beta)$ *satisfies the asymptotics*

$$z(\mu,\beta) = -\mu + d(1+\beta) - \frac{d(1+\beta^2)}{2!}\frac{1}{\mu} - (1+\beta)\big((d-1)\beta^2 + 2d\beta + d - 1\big)\frac{1}{\mu^2} + \frac{d^2\big((2d-1)\beta^2 + 4d\beta + (2d-1)\big)\big((4d+1)\beta^2 + 8d\beta + 4d + 1\big)}{(2!)^2}\frac{1}{\mu^3} + O\left(\frac{1}{\mu^4}\right),$$

as $\mu \to \infty$.

Proof. For the sufficiently large μ numbers $\left|-\left(d(1+\beta)\frac{1}{\mu} + \frac{d((2d-1)\beta^2+4d\beta+(2d-1))}{2!}\frac{1}{\mu^2} + O(\frac{1}{\mu^3})\right)\right| < 1$ holds, and the asymptotic in Lemma 6 turns to be

$$z = \frac{1}{-\frac{1}{\mu} - d(1+\beta)\frac{1}{\mu^2} - \frac{d\big((2d-1)\beta^2 + 4d\beta + (2d-1)\big)}{2!}\frac{1}{\mu^3} + O(\frac{1}{\mu^4})} =$$

$$-\mu\left(\frac{1}{1 + d(1+\beta)\frac{1}{\mu} + \frac{d((2d-1)\beta^2 + 4d\beta + (2d-1))}{2!}\frac{1}{\mu^2} + O(\frac{1}{\mu^3})}\right) =$$

$$-\mu\left(1 - \left(d(1+\beta)\frac{1}{\mu} + \frac{d\big((2d-1)\beta^2 + 4d\beta + (2d-1)\big)}{2!}\frac{1}{\mu^2} + O\left(\frac{1}{\mu^3}\right)\right) + \left(d(1+\beta)\frac{1}{\mu} + \frac{d\big((2d-1)\beta^2 + 4d\beta + (2d-1)\big)}{2!}\frac{1}{\mu^2} + O\left(\frac{1}{\mu^3}\right)\right)^2 - \left(d(1+\beta)\frac{1}{\mu} + \frac{d\big((2d-1)\beta^2 + 4d\beta + (2d-1)\big)}{2!}\frac{1}{\mu^2} + O\left(\frac{1}{\mu^3}\right)\right)^3 + \ldots\right)$$

$$= -\mu + 2d(1+\beta) - \frac{d(1+\beta^2)}{2!}\frac{1}{\mu} - d^2(1+\beta)\big((d-1)\beta^2 + 2d\beta + d - 1\big)\frac{1}{\mu^2} + \frac{d^2\big((2d-1)\beta^2 + 4d\beta + (2d-1)\big)\big((4d+1)\beta^2 + 8d\beta + 4d + 1\big)}{(2!)^2}\frac{1}{\mu^3} + O\left(\frac{1}{\mu^4}\right)$$

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