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A STUDY OF NONLOCAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR THIRD-ORDER EQUATIONS WITH MULTIPLE CHARACTERISTICS

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Abstract. *In this article, the authors studied one nonlcal nonlinear boundary value problem for a third-order nonlinear equation with multiple characteristics. The unique solvability to the problem was proven. The uniqueness of the solution to the boundary value problem was proven by the method of energy. To prove the existence of a solution to this problem, an auxiliary problem was considered. The solution of an auxiliary problem reduced the original problem to a nonlinear integral equation. Subsequently, the solvability of the nonlinear integral equation was established.*
Keywords: *nonlinearity, uniqueness, existence, nonlinear integral equation.*

I. INTRODUCTION

The following equation refers to poorly studied odd-order equations:

$$L(u) \equiv u_{xxx} - u_y = f(x, y, u, u_x, u_{xx}), \quad (MC)$$

which is called an equation with multiple characteristics (*MC*) ([1]).

The equation with multiple characteristics (MC) arises in various problems of physics and mechanics, making it of significant theoretical and applied interest.

The well-known Korteweg-de Vries equation *(KdV)*

$$u_y + uu_x + \beta u_{xxx} = 0 \qquad (KdV)$$

which is the object of research by many authors and occupies an important place in the study of nonlinear wave propagation in weakly dispersive media ([2 - 5]).

The KdV equation finds applications in diverse fields, such as fluid dynamics (e.g., modeling gravitational waves in shallow water and nonlinear Rossby waves), plasma physics (e.g., describing ion-acoustic waves), electrical engineering (e.g., analyzing nonlinear circuits), and even epidemiology (e.g., simulating the time evolution of infected individuals during an epidemic), etc. ([2 – 6])

Some characteristic features of wave propagation in dispersive media can be traced already in the linear approximation ([5]):

$$u_y + \beta u_{xxx} = 0. \qquad (LKdV)$$

The *(LKdV)* equation describes sufficiently long waves in media where the limit $\frac{\omega}{k}$ (phase velocity) as $k \to 0$ has a finite value (weakly dispersive waves). The *(LKdV)* equation is called the linearized Korteweg-de Vries equation [2 - 5].

II. LITERATURE ANALYSIS

The study of boundary value problems for third-order equations with multiple characteristics is relevant to this and other practical applications of the above.

The first paper addressing the initial-boundary value problem of the Korteweg-de Vries (KdV) equation on the finite interval (0,1) was by Bubnov in 1979 [10], who considered the initial-boundary value problem with general boundary conditions. Since then, numerous authors have worked on improving existing results and presenting new findings in recent years. ([7-9])

Note that some linear boundary value problems for a linear equation with multiple characteristics of a third-order were considered in [1, 11 - 12].

Reference [11] investigates a linear boundary value problem for a third-order nonlinear equation with multiple characteristics. A nonlinear boundary value problem for a third-order linear equation with multiple characteristics was analyzed in [7]. Works [13 - 15] investigate the unique solvability of a nonlinear boundary value problem for a nonlinear third-order equation with multiple characteristics in a curvilinear domain. In each of the above studies, the boundary conditions share a linear relationship.

A nonlocal linear boundary value problem for a third-order linear equation with multiple characteristics was studied in [16 - 18].

This article addresses a nonlinear nonlocal boundary value problem involving a third-order linear equation with multiple characteristics.

III. RESULTS

STATEMENT OF THE PROBLEM

PROBLEM A. It is required to determine in domain $D=\{(x,y):0<x<1,\, 0<y\le 1\}$ function $u(x,y)$ that has the following properties:

1) $u(x,y)\in C^{3,1}_{x,y}(D)\cap C^{2,0}_{x,y}(\overline{D})$;

2) which is a regular solution to the following equation:

$$L(u)\equiv u_{xxx}-u_y=f(x,y,u(x,y)) \qquad (1)$$

in domain D;

3) satisfying the following conditions

$$u(x,0)=g(x,u(x,1)),\quad 0\le x\le 1, \qquad (2)$$

$$u(0,y)=\varphi_1(y),\ 0\le y\le 1, \qquad (3)$$

$$u_x(0,y)=\varphi_2(y), 0\le y\le 1, \qquad (4)$$

$$u(1,y)=\psi(y), 0\le y\le 1, \qquad (5)$$

and matching conditions

$$g(0,u(0,0))=\varphi_1(1),\qquad \frac{\partial g(0,u(0,0))}{\partial x}=\varphi_2(1),\qquad g(1,u(1,0))=\psi(1).$$

Theorem (*the uniqueness of the solution*). Let for $g(x,u)$ be continuous functions of their arguments $0\le x\le 1$ for any $|u|<K$, satisfying the following condition

$$|g(x,u_1)-g(x,u_2)|\le l|u_1-u_2|, \qquad (6)$$

$$0<l\le \frac{1}{e}. \qquad (7)$$

Then the solution to Problem A is unique.

Proof. The uniqueness of the solution to the problem is proven by the method of energy integrals, using some elementary inequalities.

Let there be two solutions to the considered problem, u_1 and u_2. Consider their difference $w = u_1 - u_2$. With regard to w, we obtain the following problem:

$$L(w) \equiv w_{xxx} - w_y = 0 \qquad (1_0)$$

$$w(x,0) = g(x,u_1(x,1)) - g(x,u_2(x,1)), \quad h_1(y) \le x \le h_2(y), \qquad (2_0)$$

$$w(0,y) = 0, \quad 0 \le y \le 1, \qquad (3_0)$$

$$w_x(0,y) = 0, 0 \le y \le 1, \qquad (4_0)$$

$$w(0,y) = 0, \quad 0 \le y \le 1. \qquad (5_0)$$

Let us prove that $w(x,y) \equiv 0$.

Having integrated the following identity

$$vwL(w) \equiv vw(w_{xxx} - w_y) = 0. \qquad (8)$$

over domain D, where $v = \exp(-x-2y)$, taking into account boundary conditions (2_0) - (5_0), we have

$$-\frac{1}{2}\int_0^1 vw_x^2\Big|_{x=1} dy - \frac{3}{2}\iint_D v\, w_x^2 dxdy - \frac{1}{2}\iint_D (v_{xxx} - v_y) w^2 dxdy$$
$$-\frac{1}{2}\int_0^1 vw^2\Big|^{y=1} dx + \frac{1}{2}\int_0^1 vw^2\Big|_{y=0} dx = 0. \qquad (9)$$

We introduce the following notation:

$$I = \frac{1}{2}\int_0^1 vw_x^2\Big|_{x=1} dy + \frac{3}{2}\iint_D v\, w_x^2 dxdy + \frac{1}{2}\iint_D vw^2 dxdy \ge 0. \qquad (10)$$

According to notation (10) from (9), we have

$$I = -\frac{1}{2}\int_0^1 vw^2\Big|^{y=1} dx + \frac{1}{2}\int_0^1 vw^2\Big|_{y=0} dx.$$

With condition (6), we have

$$I \le \frac{1}{2}\int_0^1 (l^2 - e^{-2}) w^2 v\Big|_{x=1} dx \quad .$$

When condition (7) be satisfied, we arrive at the following inequality $I \le 0$. Hence, $I = 0$.

Then from (10) we obtain the following conditions:

$$\begin{cases} w_x(1,y) = 0 \;, 0 \le y \le 1, \\ w_x(x,y) = 0, \quad (x,y) \in D, \\ w(x,y) = 0, \quad (x,y) \in D. \end{cases}$$

Hence, we have $w(x,y) = p(y), (x,y) \in D$.

Since $w(h_2(y),y) = 0$, $0 \le y \le 1$, then $p(y) \equiv 0$.

Due to continuity $w(x,y)$ in $\overline{D}$ we have $w(x,y) = 0$.

EXISTENCE OF THE SOLUTION

Before proceeding to the proof of the existence of a solution to Problem A, it is necessary to study the following auxiliary problem.

PROBLEM B. It is required to determine in domain D a regular solution $u(x,y)\in C^{3,1}_{x,y}(D)\cap C^{2,0}_{x,y}(\overline{D})$ to equation

$$L(u)\equiv u_{xxx}-u_y=f(x,y) \qquad (1_1)$$

satisfying the following conditions

$$u(x,0)=\varphi_0(x),\quad 0\le x\le 1, \qquad (2_0)$$

$$u(0,y)=\varphi_1(y), 0\le y\le 1, \qquad (3)$$

$$u_x(0,y)=\varphi(y), 0\le y\le 1, \qquad (4)$$

$$u(1,y)=\psi(y), 0\le y\le 1, \qquad (5)$$

and matching conditions

$$\varphi_1(0)=\varphi_0(0),\quad \varphi_2(0)=\varphi_0'(0),\quad \psi(0)=\varphi_0(1).$$

The Green's function for Problem B, constructed in [5], is given by:

$$u(x,y)=-\frac{1}{\pi}\int_0^y G_{\xi\xi}(x,y;0,\eta)u(0,\eta)d\eta-\frac{1}{\pi}\int_0^y G_{\xi}(x,y;0,\eta)u_{\xi}(0,\eta)d\eta-$$

$$-\frac{1}{\pi}\int_0^y G_{\xi\xi}(x,y;1,\eta)u(1,\eta)d\eta+\frac{1}{\pi}\int_0^1 G(x,y;\xi,0)u(\xi,0)d\xi-\frac{1}{\pi}\iint_D G(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta,$$

(11)

where $G(x,y;\xi,\eta)=U(x,y;\xi,\eta)-W(x,y;\xi,\eta)$.

Formula (11) gives a solution to problem B.

Function $G(x,y;\xi,\eta)$ we call the Green function of problem B.

Note that for function $G(x,y;\xi,\eta)$ the same estimates as for function $U(x,y;\xi,\eta)$ are true ([1]).

$$U(x.y;\xi,\eta)=\begin{cases}(y-\eta)^{-\frac{1}{3}}f\left(\dfrac{x-\xi}{(y-\eta)^{\frac{1}{3}}}\right), & x\ne\xi,\ y>\eta,\\ 0, & y\le\eta.\end{cases}$$

$$V(x.y;\xi,\eta)=\begin{cases}(y-\eta)^{-\frac{1}{3}}\varphi\left(\dfrac{x-\xi}{(y-\eta)^{\frac{1}{3}}}\right), & x>\xi,\ y>\eta,\\ 0, & y\le\eta.\end{cases} \qquad (12)$$

Let's define the following functions:

$$f(t)=\int_0^{+\infty}\cos(\lambda^3-\lambda t)d\lambda,\ -\infty<t<+\infty,$$

$$\varphi(t)=\int_0^{\infty}\left[\exp\left(-\lambda^3-\lambda t\right)+\sin\left(\lambda^3-\lambda t\right)\right]d\lambda,\quad t=\frac{x-\xi}{(y-\eta)^{1/3}}.$$

Functions $f(t)$, $\varphi(t)$ called Airy functions, satisfy the following equation ([1]):

$$z''(t)+\frac{1}{3}tz(t)=0.$$

The following relations hold:

$$\int_0^{+\infty} f(t)dt = \frac{2}{3}\pi,\ \int_0^{+\infty} \varphi(t)dt = 0,\ \int_{-\infty}^{0} f(t)dt = \frac{\pi}{3}. \tag{13}$$

The following relations are true for functions $U(x,y;\xi,\eta),\ V(x,y;\xi,\eta)$:

$$\left|U(x,y;\xi,\eta)\right| < \frac{C}{(y-\eta)^{1/3}}$$

$$\left.\begin{array}{l}\left|\dfrac{\partial^{i+j}U(x,y;\xi,\eta)}{\partial x^i \partial y^j}\right| \\ \left|\dfrac{\partial^{i+j}V(x,y;\xi,\eta)}{\partial x^i \partial y^j}\right|\end{array}\right\} < C_1 \frac{|x-\xi|^{\frac{2i+6j-1}{4}}}{|y-\eta|^{\frac{2i+6j-1}{4}}}, \tag{14}$$

as $\dfrac{x-\xi}{(y-\eta)^{1/3}} \to +\infty,\ i+j \ge 1,\ C>0,\ C_1>0,$

$$\left|\frac{\partial^{i+j}U(x,y;\xi,\eta)}{\partial x^i \partial y^j}\right| < \frac{C_2}{|y-\eta|^{\frac{i+3j+1}{4}}} \exp\left(-C_3 \frac{|x-\xi|^{\frac{3}{2}}}{|y-\eta|^{\frac{1}{2}}}\right), \tag{15}$$

as $\dfrac{x-\xi}{(y-\eta)^{1/3}} \to -\infty,\ i+j \ge 1,\ C_2>0,\ C_3>0.$

THEOREM(***the existence of the solution***). Let, along with the conditions of the uniqueness theorem, the following conditions be satisfied:

$$\varphi_1(y) \in C^2[0,1],\ \varphi_2(y) \in C^1[0,1],\ \psi(y) \in C[0,1],$$

and for $x \in [0;1]$ for any $|u| < K$ the following inequalities hold:

$$|g(x,\ p)| < N\ .$$

Then the solution to problem (1) - (5) exists.

Proof. We will have to demonstrate that a solution exists.

From (11), we find

$$u(x,y) = -\frac{1}{\pi}\int_0^y G_{\xi\xi}(x,y;0,\eta)\varphi_1(\eta)d\eta - \frac{1}{\pi}\int_0^y G_{\xi}(x,y;0,\eta)\varphi_2(\eta)d\eta -$$
$$-\frac{1}{\pi}\int_0^y G_{\xi\xi}(x,y;1,\eta)\psi(\eta)d\eta + \frac{1}{\pi}\int_0^1 G(x,y;\xi,0)g(\xi,\nu(\xi))d\xi - \frac{1}{\pi}\iint_D G(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta\ , \tag{16}$$

where $u(x,0) = \nu(x)$.

Passing to the limit for $y \to 1$ in (16), we obtain:

$$\nu(x) = \frac{1}{\pi}\int_0^1 G(x,1;\xi,0)g(\xi,\ \nu(\xi))d\xi + F(x)\ , \tag{17}$$

where

$$F(x)=-\frac{1}{\pi}\int_0^y G_{\xi\xi}(x,1;0,\eta)\varphi_1(\eta)d\eta-\frac{1}{\pi}\int_0^y G_{\xi}(x,1;0,\eta)\varphi_2(\eta)d\eta-$$
$$-\frac{1}{\pi}\int_0^y G_{\xi\xi}(x,1;1,\eta)\psi(\eta)d\eta-\frac{1}{\pi}\iint_D G(x,1;\xi,\eta)f(\xi,\eta)d\xi d\eta\ , \qquad (18)$$

(17) is a nonlinear Fredholm integral equations with respect $\nu(x)$.

We will prove the unique solvability of this system using the method of successive approximations.

Let there exist numbers K and N such that

$$|F(x)|\le K-1,\ |G(x,\ \nu(x))|\le N,\ |\nu(x)|\le K. \qquad (19)$$

Assuming that

$$\left.\begin{aligned}&\nu^{(0)}(x)=F(x)\le|F(x)|\le K-1<K,\\&\nu^{(n)}(x)=F(x)+\frac{1}{\pi}\int_0^1 G(x,1;\xi,0)g(\nu^{(n-1)}(\xi))d\xi.\end{aligned}\right\} \qquad (20)$$

Assuming $n=1$ in equation (20) and using the estimate (14), we have

$$|\nu^{(1)}(x)|<K-1+\frac{2N}{\pi}\int_0^1 d\xi=K-1+\frac{2N}{\pi}$$

In order for xϵ[0,1] the value of the argument ν(x) of the function g(ν(x)) to satisfy

Inequality (20), under which f is limited, it is necessary to fulfill the conditions

$$K-1+\frac{2N}{\pi}<K\ , \qquad \text{or} \qquad N<\frac{\pi}{2}. \qquad (21)$$

For n = 2 from (36) we find

$$|\nu^{(2)}(X)|<K-1+\frac{2N}{\pi}.$$

According to (21)

$$|\nu^{(2)}(x)|<K$$

that is, when repeating (in the second) approximation of the argument τ(y) , the function g(τ(y),y) does not leave the limited region (19).

Applying the complete method of induction, we conclude that none of the successive approximations will leave the region (19) if condition (21) are met.

Now, we will show that the limit of the sequence $\{\tau^{(n)}(y)\}$ exist.

To establish this, it suffices to prove the convergence of the series

$$\nu^{(0)}(x)+\left(\nu^{(1)}(x)-\nu^{(0)}(x)\right)+\left(\nu^{(2)}(x)-\nu^{(1)}(x)\right)+\ldots+\left(\nu^{(k)}(x)-\nu^{(k-1)}(x)\right)+\ldots \qquad (22)$$

Let's estimate the absolute values of the terms in series (22).

$$\left.\begin{array}{l} \left|\nu^{(0)}\right| \le K-1 < K, \\ \left|\nu^{(1)}-\nu^{(0)}\right| < \dfrac{2N}{\pi} < 1, \\ \left|\nu^{(2)}-\nu^{(1)}\right| < \dfrac{2^2 lN}{\pi^2} < \dfrac{2l}{\pi} < \dfrac{2}{\pi e} \\ \left|\nu^{(3)}-\nu^{(2)}\right| \le \dfrac{2^2 l^2}{\pi^2} < \dfrac{2^2}{e^2\pi^2}, \\ \ldots. \\ \left|\nu^{(k+1)}-\nu^{(k)}\right| < \dfrac{2^k}{e^k\pi^k}, \\ \ldots \end{array}\right\}$$

It is evident that the absolute value of each term in series (22) is no greater than the corresponding term in the power series.

$$\sum_{k=1}^{\infty}\frac{2^k}{e^k\pi^k}. \tag{23}$$

This is an infinite geometric series with a magnitude of the common ratio that is less than unity.

Since the series (23) converges uniformly, it follows that it also converges absolutely and uniformly.

Taking the limit under the integral sign in equation (20), we obtain

$$\nu(x)=\frac{1}{\pi}\int_0^1 G(x,1;\xi,0)g(\nu(\xi))d\xi + F(x).$$

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