

MATHEMATICAL MODEL AND ANSYS ANALYSIS OF THE LEAD SCREW–NUT MECHANISM OF THE RT117 MACHINE TOOL

Akhmedov Azamat Khaitovich

Professor of Tashkent State Technical University named after Islam Karimov, doctor of technical sciences

+998901232287

azam0602@mail.ru

Khojibekov Tokhir Dilmuratovich

Independent Researcher of Tashkent State Technical University named after Islam Karimov

+998998027136

hojibek2016@gmail.com

Abdoollayev Javohir Sharofovich

Student at Target International School

abd.sharof@gmail.com

+998935633947

Abstract – This article presents a mathematical model and a sequential ANSYS Mechanical 2024 R2 analysis of the lead screw–nut drive of the RT117 machine tool (Rigid Dynamics followed by Static Structural). The relevance of the work is determined by the need to estimate inertial loads and the elastic response of a long, slender screw in machine-tool feed drives before experimental testing. A kinematic relation between the screw and the nut–apron system was derived, and a reduced two-coordinate Lagrangian model was constructed to interpret the numerical time series. The mechanism consisted of a steel screw 60 mm in diameter and 2000 mm long, a nut and a 400 kg apron, with a total translating mass of 446.849 kg. A constant angular velocity of 144 rad/s, a lead of 9 mm per revolution and a cam displacement law of 59 points over 1.12 s were specified. The most loaded dynamic instant was transferred to an elastic model of the screw by the Motion Loads procedure, and Inertia Relief was used to balance the resultant axial load. The peak cam force was 21.163 kN, the peak force transmitted through the nut was 19.169 kN, and the largest torque modulus in the motor joint was 27.457 N·m. In the elastic stage the maximum von Mises stress was 17.597 MPa and the maximum total deformation was 0.0379 mm. The analytical relations agreed with the ANSYS results at all 1169 time points up to numerical rounding. Thread friction, working load and experimental verification remain for further study.

Keywords: lead screw, nut, machine tool feed drive, ANSYS Mechanical, Rigid Dynamics, Motion Loads, Lagrange equations, inertia relief, von Mises stress, finite element method

INTRODUCTION

Feed drives determine the positioning accuracy, productivity and service life of a machine tool, and the lead screw–nut pair is one of their most heavily loaded components. During the acceleration and braking of a heavy carriage, the inertial forces of the moving masses are transmitted to the screw, which is long and slender and

therefore deforms elastically under axial forces and torque. At the design stage it is necessary to know these loads, together with the resulting stresses and deformations, before a prototype is built. Multibody dynamics and the finite element method make such estimates possible, but the rigid-body model of the motion and the elastic model of the screw are often treated separately, so the link between the prescribed motion law and the stress state of the screw is not always explicit. This is the scientific problem addressed in the article.

Lead screw–nut mechanisms convert the rotation of a screw into the translation of a nut and of the apron attached to it, and are widely used in machine-tool feed drives. The loads in such a drive depend on the motion law, the translating masses and the lead of the screw, while the elastic response of a long, slender screw depends on how these loads are applied to it. It is therefore convenient to solve the problem in two stages: the motion and joint loads are first determined with a rigid-body model, and the most loaded instant is then transferred to an elastic model of the screw [1–4].

The RT117 mechanism contains a steel screw 60 mm in nominal diameter and 2000 mm long, a nut 120 mm long and an apron of 400 kg; the lead of the screw joint is 9 mm per revolution. The motor rotates the screw at a constant angular velocity of 144 rad/s, and an additional axial displacement of the screw is prescribed by a cam according to a tabulated law of 59 points over 1.12 s. The aim of this study is to develop a mathematical model of the mechanism, to determine the inertial loads, the joint torque and the mechanical power for this motion law, and to evaluate the stress and deformation of the screw at the most loaded instant. The calculations were performed in ANSYS Mechanical 2024 R2 with the Rigid Dynamics and Static Structural systems applied in sequence.

The object of the research is the lead screw–nut mechanism of the RT117 machine tool, and the subject is its inertial loads, joint torque, mechanical power and the stress–strain state of the screw under the prescribed motion law. To achieve the aim, the following objectives were set: (i) to derive the kinematic relation between the screw and the nut–apron system; (ii) to construct a reduced Lagrangian model and to compare it with the Rigid Dynamics results at every stored time point; (iii) to determine the peak forces, the torque and the power of the mechanism; and (iv) to transfer the most loaded instant to an elastic model of the screw and to evaluate its stress and deformation. The hypothesis is that the relations of the reduced model reproduce the results of the Rigid Dynamics solver up to numerical rounding, so that the loads transferred to the elastic model are determined consistently.

The scientific novelty consists in following one prescribed motion law through a single verified sequence: from the kinematic coupling of the screw and the nut, through the inertial loads and the joint torque, to the equivalent stress and deformation of the screw. The same loads that were verified analytically were applied to the elastic model, so the stress state is related directly to the parameters of the drive, namely the lead, the translating masses and the motion law.

LITERATURE REVIEW

The design of power screws and the analysis of the forces transmitted through

them are described in classical machine design textbooks [1], while the kinematics of mechanisms and the motion of cam-driven systems are treated in the theory of machines and mechanisms [2]. These sources give the basic relations between the lead of a screw, the axial force and the driving torque, but they are oriented mainly toward steady operation and static load cases and do not follow how a prescribed motion law of a heavy carriage determines the time history of the loads.

The dynamics of systems of rigid and flexible bodies connected by kinematic joints is presented in the multibody dynamics literature, where the motion is described by Lagrange equations with constraints [3]. This approach is implemented in commercial solvers: in ANSYS Mechanical, the motion and the joint loads found in a Rigid Dynamics analysis can be transferred to a flexible body by the Motion Loads procedure [4], which makes it possible to combine the two levels of modelling in one sequence.

The elastic response of the screw is described by the theory of elasticity [5], whose small-deformation relations and Hooke's law underlie the finite element method [6, 7, 10]. The monographs on this method discuss element formulations and the direct solution of the resulting equations, and the strength of a material under a multiaxial stress state is usually assessed through an equivalent stress [9]. For a body that is not fixed in space, the equilibrium of the resultant load can be provided by the Inertia Relief option described in the solver theory reference [8].

A critical comparison shows that each group of sources covers one part of the problem: machine design textbooks give simplified force relations, multibody dynamics gives the motion and the joint loads, and the theories of elasticity and finite elements give the stress state. The available works are mainly general textbooks and software manuals; they do not present a worked and verified sequence for a specific feed drive in which one motion law is followed from the kinematic coupling of the screw and the nut, through the inertial loads and the joint torque, to the stress and deformation of the screw. This unexplored gap is filled by the present article, which combines an analytical model verified at every time point with a sequential Rigid Dynamics – Static Structural analysis.

METHODOLOGY

The geometric and mass parameters of the model are given in Table 1. In the dynamic stage the screw and the nut were rigid bodies; the 400 kg apron was a point mass attached to the nut by a remote attachment with zero own moments of inertia, so only its translational inertia is considered. The material was structural steel (St45 in the calculation report) with $E = 200$ GPa, $\nu = 0.30$ and $\rho = 7850$ kg/m³, which gives $m_s = \rho V_s = 42.114$ kg. Thread friction, torsional stiffness and joint damping were set to zero and no cutting force was applied, so the loads below are purely inertial loads produced by the prescribed motion.

Table 1.

Parameter	Value
Nominal diameter / length of the screw	60 mm / 2000 mm
Axial length of the nut	120 mm
Lead of the screw joint, h	9 mm/rev

Screw mass, m_s	42.114 kg
Nut mass, m_n	4.735 kg
Apron mass, m_f	400 kg
Mass of nut and apron, $m_N = m_n + m_f$	404.735 kg
Total translating mass, $m_{\Sigma} = m_s + m_N$	446.849 kg
Moment of inertia of the screw about its axis, J_s	0.018095 kg·m ²

The global X axis coincided with the screw axis. The screw was connected to the base by a cylindrical joint, which left only translation along and rotation about the axis [2], and to the nut by a screw joint in which one turn corresponded to a relative displacement $h = 9$ mm [1]. The rotation of the nut was constrained, so the rotation of the screw was converted into an axial displacement of the nut. Let u_s be the displacement of the screw, u_n that of the nut and apron, and θ the rotation angle of the screw. Then

$$k = \frac{h}{2\pi} = \frac{0.009}{2\pi} = 1.4324 \times 10^{-3}, \quad (1)$$

$$u_n(t) = u_s(t) - k\theta(t), \quad (2)$$

where k is in m/rad. For $\theta(t) = \omega t$ the time derivatives give

$$v_n = v_s - k\omega, \quad a_n = a_s. \quad (3)$$

At $\omega = 144$ rad/s the nut velocity differs from the screw velocity by the constant $k\omega = 0.206265$ m/s (1375.1 rev/min), and the axial accelerations are equal. At the end of the 1.12 s interval $k\theta = 231.017$ mm, so although the screw returns close to its initial position, the nut remains displaced along the axis.

To interpret the ANSYS time series analytically, a reduced model with generalized coordinates $q = (u_s, \theta)$ was derived. With $m_N = m_n + m_f = 404.735$ kg and the coupling (2), the kinetic energy is

$$T = \frac{1}{2}m_s\dot{u}_s^2 + \frac{1}{2}m_N(\dot{u}_s - k\dot{\theta})^2 + \frac{1}{2}J_s\dot{\theta}^2. \quad (4)$$

Elastic and damping terms are not introduced for the axial motion, and the Lagrange equations of the second kind [3]

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i, \quad (5)$$

with the generalized forces $Q = (F_c, M_m)$ give

$$(m_s + m_N)\ddot{u}_s - km_N\ddot{\theta} = F_c, \quad (6)$$

$$-km_N\ddot{u}_s + (J_s + k^2m_N)\ddot{\theta} = M_m. \quad (7)$$

Here F_c is the cam force and M_m the torque in the motor joint; $m_s + m_N = 446.849$ kg, $km_N = 0.57974$ kg·m and $J_s + k^2m_N = 0.018925$ kg·m², with $J_s = 0.018095$ kg·m². This model serves only to explain and verify the results; the Rigid Dynamics solver itself solves the general multibody system with constraint equations [3]. For a constant angular velocity ($\ddot{\theta} = 0$) Eqs. (6) and (7) reduce to

$$F_n = m_N a_s, \quad F_c = (m_s + m_N) a_s, \quad (8)$$

$$M_m = -kF_n = -\frac{h}{2\pi} F_n, \quad (9)$$

where F_n is the axial force transmitted through the nut and $F_c - F_n = m_s a_s$ is the inertial force of the screw itself. The motor-joint power is $P_m = M_m \omega$, and the translational kinetic energy and the power balance are

$$T_{tr} = \frac{1}{2} m_s v_s^2 + \frac{1}{2} m_N v_n^2, \quad P_\Sigma = F_c v_s + M_m \omega = \frac{dT_{tr}}{dt}. \quad (10)$$

The cam motion was entered as a displacement joint load from the table of 59 points, processed by ANSYS with Program Controlled fitting and 200 segments; the interpolation order is not stated in the report. The analysis interval was 0–1.12 s with a maximum time step of 0.001 s, 1169 time points were stored, and two bodies and two joints (no contacts) were active. Displacements, velocities, accelerations, forces and torques were taken from joint probes on a common time grid.

ANALYSIS AND RESULTS

Table 2 gives the extreme values of the kinematic parameters, forces and torque. The screw displacement ranges from –1.5604 to 28.3395 mm (29.8999 mm in total), whereas the nut and apron reach –231.0165 mm; the difference is the term $-k\omega t$ of Eq. (2). The screw velocity changes sign in the return part, while the nut velocity is negative throughout, with a largest modulus of 0.541110 m/s; their difference is constant at 0.206265 m/s. The axial accelerations of the screw and the nut coincide, with a minimum of –39.1969 m/s² at 0.967789 s and a maximum of +47.3609 m/s² at 1.079310 s. These peaks are derivatives of the prescribed motion law and are not interpreted as contact impacts, since no impact model was introduced.

Table 2.

Quantity	Min.	Time, s	Max.	Time, s
Screw displacement u_s , mm	–1.5604	1.07858	28.3395	0.71800
Nut and apron displacement u_n , mm	–231.0165	1.12000	0.0000	0.00000
Screw velocity v_s , m/s	–0.334846	1.06358	0.051825	0.38400
Nut and apron velocity v_n , m/s	–0.541110	1.06358	–0.154440	0.38400
Axial acceleration $a_s = a_n$, m/s ²	–39.1969	0.96779	+47.3609	1.07931
Cam force F_c , kN	–17.5151	0.96779	21.1632	1.07931
Nut force F_n , kN	–15.8643	0.96779	19.1686	1.07931
Motor-joint torque M_m , N·m	–27.457	1.07931	+22.724	0.96779

The cam force depends on the total translating mass of 446.849 kg and the nut force on the mass of 404.735 kg of the nut and apron. At the positive peak $F_c = 21.163$ kN and $F_n = 19.169$ kN; their difference of 1.995 kN is the inertial force of the screw. According to Eq. (9) the torque is proportional to the axial force and to the lead. The largest torque modulus is 27.457 N·m and the corresponding mechanical power $|M_m|\omega$ is 3.954 kW; this is the power of the inertial loads only, and thread friction, working load and drive efficiency must be added when selecting a motor.

The translational kinetic energy varies from 4.883 to 61.614 J (maximum at 1.06358 s), and the rotational energy $J_s \omega^2 / 2 = 187.604$ J is constant. The analytical relations were compared with the ANSYS values at all 1169 time points. The largest residuals are 1.28×10^{-10} m for $u_n = u_s - k\omega t$, 5.29×10^{-11} m/s for $v_n = v_s - k\omega$, 0 for $a_n = a_s$, $6.2\text{--}8.0 \times 10^{-6}$ N for the force relations, 1.05×10^{-8} N·m for $M_m = -kF_n$ and 1.68×10^{-6} W for the power balance. These very small values show that the results are consistent with the adopted ideal model and partly reflect the rounding of the reported numbers. The check evaluates only internal consistency; it does not replace a

comparison with experiments or a mesh-convergence study.

Since the positive acceleration peak exceeds the modulus of the negative one, the load case $t_c = 1.079310$ s was chosen for the elastic calculation. At this instant $u_s = -1.553$ mm, $u_n = -224.177$ mm, $v_s = +0.026989$ m/s, $v_n = -0.179275$ m/s, $a_s = +47.3609$ m/s², $F_c = 21163.17$ N, $F_n = 19168.60$ N and $M_m = -27.457$ N·m. These loads were transferred to the Static Structural system by the Motion Loads procedure [4] (Fig. 1); the applied forces differ from the dynamic values by only 0.00215 and 0.00196 N, and the torques are applied with opposite signs. The nut-connection remote point is the joint point through which the load is introduced, not the centre of mass of the nut.

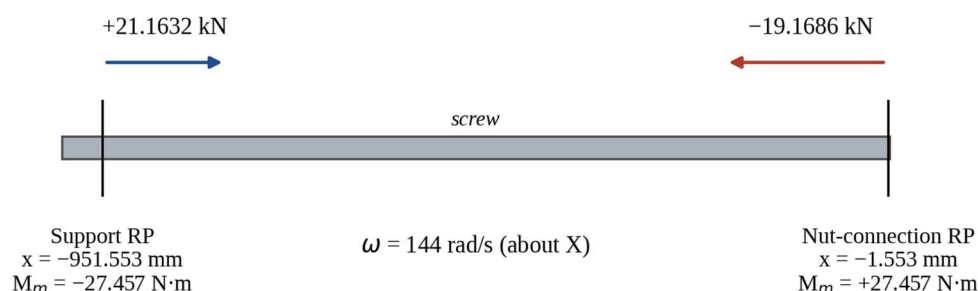


Fig. 1. Forces, torques and remote-point coordinates applied in the Static Structural model

The elastic response of the screw is described by the theory of small deformations. For the displacement vector $u = (u_x, u_y, u_z)$ the strain tensor and Hooke's law for a linear isotropic material are [5]

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (11)$$

$$\sigma_{ij} = \lambda \operatorname{tr}(\varepsilon) \delta_{ij} + 2G \varepsilon_{ij}. \quad (12)$$

where, for $E = 200$ GPa and $\nu = 0.30$, $\lambda = E\nu/[(1 + \nu)(1 - 2\nu)] = 115.385$ GPa and $G = E/[2(1 + \nu)] = 76.923$ GPa. In the finite element method $u^h = Nd^e$, $\varepsilon = Bd^e$ and $\sigma = D\varepsilon$, where N is the shape-function matrix, B the strain-displacement matrix and D the elasticity matrix; the element stiffness matrices are assembled into the global system [6]:

$$K^e = \int_{V_e} B^T DB \, dV, \quad Kd = f. \quad (13)$$

Only the screw is an active flexible body; the nut and apron are suppressed and replaced by the transferred forces and torques, so the model describes the state of the screw and not the contact pressure in the thread pair. The mesh has 165622 nodes and 94370 quadratic elements of 5 mm size; a direct solver was used [7], with Large Deflection switched off and averaged stress output. The remote points are connected to the selected faces by deformable multipoint constraints (pinball radius 72 mm at the nut connection). The exact element name is not given and no mesh-refinement series is available, so convergence is not estimated [6, 10].

The two axial forces applied to the moving screw do not sum to zero. The Inertia Relief option balances the resultant load by an acceleration and corresponding inertial

forces [8], separating the elastic displacement from the rigid-body motion. The resultant force and the equivalent acceleration are

$$F_{\Sigma,X} = F_c - F_n, \quad a_{IR,X} = \frac{F_{\Sigma,X}}{m_s}, \quad (14)$$

i.e. $F_{\Sigma,X} = 1994.576$ N and $a_{IR,X} = 47.3609$ m/s², equal to the axial acceleration of the dynamic stage. The centrifugal load of the rotation at 144 rad/s is radial ($\omega^2 r = 622.08$ m/s² at $r = 0.03$ m) and is not mixed with this axial acceleration. The residual support reaction is only 0.000823 N ($|R|/|F_c| \approx 3.89 \times 10^{-8}$), which confirms the equilibrium of the load case; it is not an estimate of a real bearing load.

The results of the elastic stage are given in Table 3. The equivalent von Mises stress represents a multiaxial stress state by a single value and is expressed through the principal stresses as [9]

$$\sigma_{eq} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}. \quad (15)$$

Table 3.

Result	Minimum	Maximum	Average
Equivalent (von Mises) stress, MPa	0.013379	17.597135	3.825743
Normal stress σ_x , MPa	-18.578069	+3.579021	-3.718706
Directional deformation u_x , mm	-0.036847	+0.001088	-0.027370
Total deformation, mm	0.00005163	0.037894	0.028071

The maximum equivalent stress is 17.597 MPa (Fig. 2, nodal averaged); its location is not given in the report and can be found with a separate probe. The normal stress σ_x ranges from -18.578 to +3.579 MPa with a mean of -3.719 MPa, i.e. compressive components prevail. That its minimum modulus exceeds the maximum von Mises stress is not a contradiction, because the von Mises stress depends only on the deviatoric part of the stress tensor and the extrema need not occur at the same point. A margin of safety cannot be determined from 17.597 MPa alone, since the yield strength of the material is required.

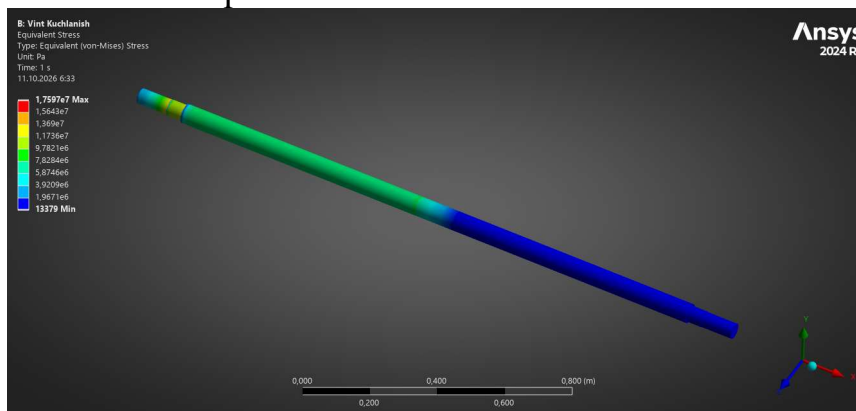


Fig. 2. Equivalent (von Mises) stress in the screw at Time = 1 s (ANSYS; unit: Pa)

The total deformation is the modulus of the displacement vector:

$$u_{tot} = \sqrt{u_x^2 + u_y^2 + u_z^2}. \quad (16)$$

Its maximum is 0.0379 mm (Fig. 3), i.e. 1.895×10^{-5} of the 2000 mm screw

length, which is consistent with the small-displacement formulation (Large Deflection off). Plastic yielding is not included in the model.

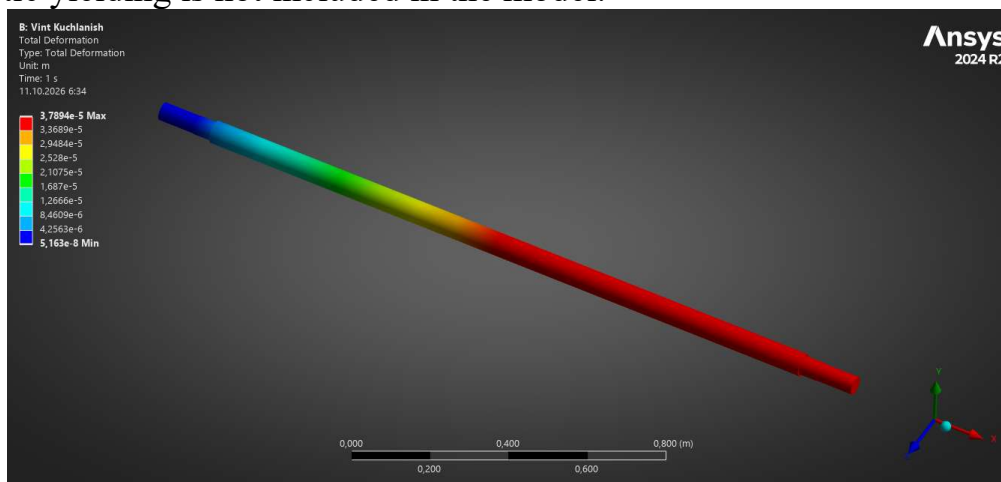


Fig. 3. Total deformation of the screw at Time = 1 s (ANSYS; unit: m)

DISCUSSION

The results answer the aim and the objectives stated in the introduction. The agreement between the reduced Lagrangian model and the Rigid Dynamics solver, with residuals of the order of 10^{-10} m for the displacement, shows that the kinematic coupling and the force relations were entered and interpreted correctly. The solver treats the general multibody system with constraint equations [3], whereas the reduced model has only two coordinates, so their agreement at all 1169 time points confirms the internal consistency of the calculation. It is not an experimental validation.

Equations (8) and (9) show that the peak loads are proportional to the axial acceleration and to the translating masses, and that the torque is proportional to the lead. The nut and the apron make up 404.735 kg of the total translating mass of 446.849 kg, i.e. about 91%, so the apron mass is the main factor of the load on the screw, while the screw itself contributes only the inertial force of 1.995 kN. Because the positive acceleration peak exceeds the negative one, the load case at 1.079310 s was a justified choice for the elastic stage.

In the elastic stage the equivalent stress of 17.597 MPa and the deformation of 0.0379 mm, which is 1.895×10^{-5} of the screw length, are small, which is consistent with the small-displacement formulation used with Large Deflection switched off [5]. The compressive normal stress exceeds the maximum von Mises stress in modulus; this is not a contradiction, because the von Mises stress depends only on the deviatoric part of the stress tensor [9]. A margin of safety cannot be determined from the stress value alone, since the yield strength of the material is also required.

The work has several limitations. Thread friction, torsional stiffness and joint damping were set to zero and no working load was applied, so the loads are purely inertial. The nut and the apron were replaced by the transferred forces and torques, so the contact pressure in the thread pair was not evaluated. A mesh-convergence study and an experiment were not performed, the location of the maximum stress was not determined, and the residual reaction of 0.000823 N in the Inertia Relief setup is not a real bearing load. These points define the directions of further research.

CONCLUSION

The following conclusions were obtained:

1. The Rigid Dynamics and Static Structural calculations were linked through load transfer, so the motion of the mechanism, the inertial loads and the elastic response of the screw were evaluated in one sequence.
2. A lead of 9 mm per revolution and 144 rad/s give a relative nut–screw velocity of 0.206265 m/s; the relation $u_n = u_s - k\omega t$ was satisfied and the accelerations were equal at all time points. The screw displacement range is 29.8999 mm, the final nut–apron displacement is –231.0165 mm, and the axial acceleration varies from –39.1969 to +47.3609 m/s²; the maximum at 1.079310 s defines the most loaded instant.
3. The peak cam and nut forces are 21.163 and 19.169 kN (masses 446.849 and 404.735 kg); their difference is the inertial force of the screw. The relation $M_m = -hF_n/(2\pi)$ was satisfied; the largest torque modulus is 27.457 N·m and the mechanical power modulus is 3.954 kW.
4. The translational kinetic energy reaches 61.614 J, and the largest residual of the power balance, 1.68×10^{-6} W, confirms the internal consistency of the calculation (not an experimental validation).
5. At the critical instant the maximum equivalent stress is 17.597 MPa, the maximum total deformation is 0.0379 mm and the minimum normal stress is –18.578 MPa. The model has 165622 nodes and 94370 quadratic elements and uses Inertia Relief; the residual reaction of 0.000823 N is not a real bearing load.
6. The results give a numerical basis for comparing the screw lead, the apron mass and the motion law. Thread friction, the working load, a mesh-convergence study and experimental verification remain to be added.

REFERENCES

1. Budynas, R. G., Nisbett, J. K. Shigley's Mechanical Engineering Design. 11th ed. New York: McGraw-Hill, 2020. <https://www.mheducation.com/highered/product/shigley-s-mechanical-engineering-design-budynas.html>
2. Norton, R. L. Design of Machinery: An Introduction to the Synthesis and Analysis of Mechanisms and Machines. 6th ed. New York: McGraw-Hill, 2020. <https://www.mheducation.com/highered/product/design-machinery-norton.html>
3. Shabana, A. A. Dynamics of Multibody Systems. 5th ed. Cambridge: Cambridge University Press, 2020. <https://doi.org/10.1017/9781108757553>
4. Ansys, Inc. Transferring Motion Loads. Ansys Mechanical User's Guide, Release 2024 R2. https://ansyshelp.ansys.com/public/Views/Secured/corp/v242/en/wb_sim/rbd_mload_transfer.html
5. Timoshenko, S. P., Goodier, J. N. Theory of Elasticity. 3rd ed. New York: McGraw-Hill, 1970. <https://books.google.com/books?vid=ISBN9780070647206>
6. Cook, R. D., Malkus, D. S., Plesha, M. E., Witt, R. J. Concepts and Applications of Finite Element Analysis. 4th ed. New York: John Wiley & Sons, 2002.

<https://www.wiley.com/en-us/Concepts+and+Applications+of+Finite+Element+Analysis%2C+4th+Edition-p-9780471356059>

7. Bathe, K. J. Finite Element Procedures. 2nd ed. Watertown, MA: Klaus-Jürgen Bathe, 2014. <https://www.bathe.org/Finite-Element-Procedures/>

8. Ansys, Inc. Inertia Relief. Ansys Mechanical APDL Theory Reference, Release 2024 R2.

https://ansyshelp.ansys.com/public/Views/Secured/corp/v242/en/ans_thry/thy_tool2.html

9. Beer, F. P., Johnston, E. R., DeWolf, J. T., Mazurek, D. F. Mechanics of Materials. 8th ed. New York: McGraw-Hill, 2020.

<https://www.mheducation.com/highered/product/mechanics-materials-beer-johnston.html>

10. Zienkiewicz, O. C., Taylor, R. L., Zhu, J. Z. The Finite Element Method: Its Basis and Fundamentals. 7th ed. Oxford: Butterworth-Heinemann, 2013.

<https://doi.org/10.1016/C2009-0-24909-9>