

BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER EQUATION WITH MULTIPLE CHARACTERISTICS

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Abstract- *This article investigates the well-posedness of a boundary value problem for a class of third-order nonlinear partial differential equations characterized by multiple characteristics. Equations of this type frequently appear in hydrodynamics, plasma physics, and acoustic wave propagation problems, where higher-order dispersion and nonlinear interactions play a decisive role.*

The primary mathematical objectives of the study are to prove the existence and uniqueness of solutions within appropriate function spaces. An energy method is employed to establish uniqueness. By constructing a suitable energy functional and deriving refined energy estimates, it is shown that the corresponding homogeneous problem possesses only the trivial solution, thereby guaranteeing uniqueness under the prescribed boundary conditions.

The method of continuation with respect to a parameter is used to prove existence. Introducing a continuous parameter—which allows the original nonlinear problem to be deformed into a simpler linear analogue with a known solution—enables the establishment of global classical solvability. This approach relies on deriving uniform a priori estimates for the family of parameterized solutions, ensuring that the set of parameters for which the problem is solvable is both open and closed within the parameter space. Furthermore, the interaction between multiple characteristics and nonlinear terms is analyzed in detail, identifying the structural conditions necessary to prevent a loss of solution smoothness. The results presented extend known fundamental existence and uniqueness theorems to a broader class of third-order nonlinear differential equations.

Keywords- *Solution, uniqueness, existence, energy integral, nonlinearity, a priori estimate, local solution, parameter, continuation, global solution.*

INTRODUCTION

Nonlinear partial differential equations (PDEs) serve as a fundamental framework for modeling complex physical phenomena across fluid dynamics, plasma physics, acoustics, and nonlinear optics. Among these, higher-order dispersive equations are particularly essential for capturing the delicate interplay between wave steepening and dispersion. While even-order parabolic and elliptic equations have been exhaustively studied, odd-order differential equations—especially those with multiple characteristics (MC)—present unique analytical challenges due to their asymmetric spectral properties, non-self-adjoint operators, and complex boundary value formulations. Early foundational efforts to address odd-order parabolic problems, such as the pioneering work by Cattabriga [1], demonstrated that boundary condition

requirements for such systems differ fundamentally from standard parabolic theory. Despite their importance, odd-order equations with multiple characteristics remain relatively understudied in modern mathematical physics.

The following equation refers to poorly studied odd-order equations:

$$L(u) \equiv u_{xxx} - u_y = f(x, y, u, u_x, u_{xx}), \quad (MC)$$

which is called an equation with multiple characteristics (MC) ([1]).

The most celebrated benchmark within the class of odd-order nonlinear dispersive models is the Korteweg–de Vries (KdV) equation (KdV)

$$u_y + uu_x + \beta u_{xxx} = 0 \quad (KdV)$$

which is the object of research by many authors and occupies an important place in the study of nonlinear wave propagation in weakly dispersive media [2 – 4].

First formulated to describe the evolution of long surface waves in shallow water channels [2], the KdV equation has grown into a cornerstone of modern nonlinear wave theory [3, 4]. It captures the fundamental mechanism by which nonlinear convective steepening is balanced by high-frequency third-order dispersion, leading to the formation of stable, localized wave packets known as solitons. Over the past several decades, the analytical reach of the KdV model has expanded well beyond classical hydrodynamics. It is extensively applied in space plasma physics to model ion-acoustic solitary waves [5], in electrical engineering to analyze signal propagation in non-uniform transmission lines, and in nonlinear lattice dynamics. Remarkably, recent interdisciplinary adaptations have even extended modified KdV frameworks to epidemiology—such as the KdV-SIR model—to simulate and analyze the wave-like time evolution of infected populations during epidemic outbreaks [6].

These equations introduce severe mathematical complications. The presence of multiple characteristics alters the domain of dependence, complicates the well-posedness of initial-boundary value problems (IBVPs), and renders traditional analytical tools—such as standard inverse scattering transforms or classical energy estimates—insufficient or non-trivial to apply. Consequently, establishing exact solutions, long-time asymptotic behavior, and structural stability for these generalized MC systems remains an open and pressing problem in partial differential equations.

LITERATURE ANALYSIS

The theoretical foundations for investigating boundary value problems (BVPs) for odd-order partial differential equations with multiple characteristics trace back to the mid XX century. In [1] established fundamental solvability results for odd-order parabolic equations, demonstrating that non-self-adjoint operators require structural asymmetries in boundary conditions—such as unequal conditions at spatial boundaries—to ensure well-posedness. Subsequent fundamental progress was made [6], who systematized the qualitative theory of BVPs for mixed and mixed-composite partial differential equations, highlighting how degenerate order and characteristic multiplicity reshape domains of dependence and energy balances.

Building upon these foundational frameworks, classical research predominantly focused on linear third-order equations with multiple characteristics [7, developed general boundary value theory for such third-order operators, proving that uniqueness and existence could be attained by combining energy integral estimates with potential methods and Green’s function formulations. While these classical results successfully resolved basic local IBVPs in standard rectangular domains, they relied heavily on linear operator linearity and symmetric assumptions, leaving nonlinear generalizations largely unexplored.

In recent years, thanks to modern developments, the analysis of odd-order equations with multiple characteristics has been extended to broader function spaces, boundary data, and operator generalizations [9,11–12, 14-15].

METHODOLOGY

To justify the well-posedness, structural stability, and uniqueness of the global solution to the boundary value problem, a functional-analytic approach was employed, combining energy estimates, non-homogeneous integral transforms, operator theory, and embedding methods in metric spaces; the approach was implemented in four successive stages.

1. Definition of the domain and problem formulation. A bounded domain with a finite time interval was defined. Taking into account the multiplicity of characteristics and the asymmetry of wave propagation—features inherent to odd-order operators—consistent initial data and asymmetric spatial boundary conditions were prescribed. All coefficients and boundary functions were assigned to appropriate classes to ensure the regularity of the classical solution.

2. Proof of uniqueness via the energy integral method. The problem was reduced to a homogeneous form by considering the difference between two hypothetical solutions. Integrating the product of the differential equation and a suitable integrating factor over the domain in question, and applying integration by parts, yielded boundary integral identities. The strict positivity and definiteness of the resulting quadratic form confirmed the uniqueness of the trivial solution, thereby justifying the well-posedness of the problem for the homogeneous system.

3. Construction of a metric space. To analyze an equivalent representation in the form of an integral operator, a complete metric space (a Banach space) was defined. The introduced metric constrains the potential growth of quantities during high-order differentiation with respect to spatial variables, ensuring uniform estimates and operator compactness.

4. Parameter continuation method and global solvability. Global existence of the solution was proved using the parameter continuation method (homotopy deformation) by introducing a parameter that connects a simplified base operator to the original operator of the boundary value problem. Uniform a priori estimates obtained via the energy method guaranteed that the solution trajectories remain bounded. Combining continuation principles with topological degree theory made it possible to extend the local solution to the entire interval, thereby confirming the global solvability of the problem.

ANALYSIS AND RESULTS

STATEMENT OF THE PROBLEM

PROBLEM A. It is required to determine in domain $D = \{(x, y) : 0 < x < 1, 0 < y \leq 1\}$ function $u(x, y)$ that has the following properties:

1) $u(x, y) \in C_{x,y}^{3,1}(D) \cap C_{x,y}^{2,0}(\overline{D})$;

2) which is a regular solution in domain D to the following equation:

$$L(u) \equiv u_{xxx} - u_y = f(x, y, u, u_x) \quad (1)$$

3) satisfying the following conditions

$$u(x, 0) = 0, \quad 0 \leq x \leq 1, \quad (2)$$

$$u(0, y) = \phi_1(y), \quad 0 \leq y \leq 1, \quad (3)$$

$$u_x(0, y) = \phi_2(y), \quad 0 \leq y \leq 1, \quad (4)$$

$$u(1, y) = \psi_1(y), \quad 0 \leq y \leq 1, \quad (5)$$

and matching conditions $\phi_1(0) = \phi_2(0) = \psi(0) = 0$.

UNIQUENESS OF THE SOLUTION

THEOREM (the uniqueness of the solution). Let $f(x, y, u, u_x)$ be continuous function of their arguments and for any $|u| < \infty, |u_x| < \infty$ satisfying the following condition

$$|f(x, y, u_1, u_{1x}) - f(x, y, u_2, u_{2x})| \leq L \{|u_1 - u_2| + |u_{1x} - u_{2x}|\}, \quad (6)$$

$$0 < L \leq \frac{1}{3}. \quad (7)$$

where L - some constant.

Then the solution to Problem A is unique.

Proof. Let there be two solutions to the considered problem, u_1 and u_2 . Consider their difference $w = u_1 - u_2$. For w , we obtain a homogeneous boundary value problem for the equation

$$L(w) \equiv w_{xxx} - w_y = f(x, y, u_1, u_{1x}) - f(x, y, u_2, u_{2x}), \quad (10)$$

Let us prove that $w(x, y) \equiv 0$.

Having integrated the following identity

$$v w L(w) \equiv v w (w_{xxx} - w_y) = v w (f(x, y, u_1, u_{1x}) - f(x, y, u_2, u_{2x})) \quad (8)$$

over domain D , where $v = \exp(-x - 2y)$, taking into account a homogeneous boundary conditions, we introduce the following notation:

$$I = \frac{1}{2} \int_0^1 v w_x^2 \Big|_{x=1} dy + \frac{1}{2} \int_0^1 v w^2 \Big|_{y=1} dx + \frac{3}{2} \iint_D v w^2 dx dy \geq 0. \quad (9)$$

According to notation (9), alternatively we have

$$I = \frac{1}{2} \iint_D (v_y - v_{xxx}) w^2 dx dy - \iint_D (f(x, y, u_1, u_{1x}) - f(x, y, u_2, u_{2x})) w v dx dy$$

In accordance with condition (6) and using Young's inequality we obtain the

following

$$I \leq \frac{1}{2} \iint_D (3L - 1)w^2 v dx dy + \frac{L}{2} \iint_D \omega_x^2 v dx dy$$

When condition (7) are satisfied, we obtain

$$\begin{aligned} \frac{1}{2} \int_0^1 v w_x^2|_{x=1} dy + \frac{1}{2} \int_0^1 v w^2|_{y=1} dx + \frac{1}{2} \iint_D (3 - L) v w_x^2 dx dy \\ + \frac{1}{2} \iint_D (1 - 3L) v w^2 dx dy = 0. \end{aligned}$$

from that we obtain the following conditions:

$$\begin{cases} w_x(1, y) = 0, & 0 \leq y \leq 1, \\ w(x, 1) = 0, & 0 \leq x \leq 1, \\ w_x(x, y) = 0, & (x, y) \in D, \\ w(x, y) = 0, & (x, y) \in D. \end{cases}$$

So $w(x, y) = 0, (x, y) \in D$.

EXISTENCE OF THE SOLUTION

THEOREM (the existence of the solution). Suppose that, in addition to the conditions of the uniqueness theorem, the following conditions are also satisfied:

$$\begin{aligned} |f(x, y, u, u_x)| \leq M, |f_y(x, y, u, u_x)| \leq M_1, |f_u(x, y, u, u_x)| \leq \\ M_2, |f_{u_x}(x, y, u, u_x)| \leq M_3, \\ \phi_1(y) \in C^1[0, 1], \phi_2(y) \in C^1[0, 1], \psi(y) \in C^1[0, 1]. \end{aligned}$$

Then the solution to Problem A is existence.

Proof. We employ the parameter continuation method. Let λ be a number in the interval $[0, 1]$. Consider the family of boundary value problems: find a function $u(x, t)$ that is a solution to the equation

$$L_\lambda u \equiv u_y - u_{xxx} = \lambda f(x, y, u, u_x) \quad (12)$$

in the domain D , and satisfies conditions (2)–(5).

Let us denote by the set of those numbers λ from the interval $[0, 1]$ for which the boundary value problem (10), (2)–(5) has a solution for a fixed ε . If the set is non-empty, open, and closed, then, as is known [10], it coincides with the entire interval $[0, 1]$.

The set is non-empty, since the number zero belongs to it. This follows from the fact that, for $\lambda=0$, problem (10), (2)–(5) was investigated in [1].

To prove that the set is open and closed, it suffices to show that, for all possible solutions $u(x, y)$ of the boundary value problem (10), (2)–(5), an a priori estimate uniform in λ holds.

We will show that the required estimate indeed exists.

Constructing an auxiliary function $\Phi(x, y)$ that satisfies the boundary conditions

$$\Phi(x, y) = \phi_1(y) + x\phi_2(y) + x^2(\psi_1(y) - \phi_1(y) - \phi_2(y)).$$

We make the change of variable: $v(x, y) = u(x, y) - \Phi(x, y)$.

Then $v(x, y)$ satisfies the homogeneous boundary conditions

$$v(x, 0) = 0, \quad 0 \leq x \leq 1,$$

$$v(0, y) = 0, \quad v_x(0, y) = 0, \quad v(1, y) = 0, \quad 0 \leq y \leq 1.$$

The equation for $v(x, y)$ takes the form

$$v_y - v_{xxx}(x, y) = \lambda f(x, y, v + \Phi, v_x + \Phi_x) - \Phi_y = F(\lambda, x, y, v, v_x) \quad (13)$$

Estimation of the nonlinearity for the function v , taking into account the properties of Φ :

$$|F(x, y, v, v_x)| \leq L(|v| + |v_x|) + F_0(x, y),$$

where $F_0(x, y) = L(|\Phi| + |\Phi_x|) + |\Phi_y| \leq M_\Phi$.

We multiply both sides of the equation (13) by $(1 - x)v(x, y)$ and integrate with respect to x

$$\int_0^1 (1 - x)vv_y dx - \int_0^1 (1 - x)vv_{xxx} dx = \int_0^1 (1 - x)vF dx.$$

Using the homogeneous boundary conditions and integrating over $y \in [0, 1]$ gives the simplified energy equality:

$$\frac{1}{2} \int_0^1 (1 - x)v^2 dx + \frac{3}{2} \iint_D (1 - x)v^2_x dx dy = \lambda \iint_D (1 - x)vF dx dy.$$

$$|F(x, y, v, v_x)| \leq L(|v| + |v_x|) + F_0(x, y) \leq L(|v| + |v_x|) + M_\Phi.$$

Consider the right-hand side of the energy identity

$$R = \iint_D (1 - x)vF dx dy$$

Using the given structural condition $|F(x, y, v, v_x)| \leq L(|v| + |v_x|) + M_\Phi$ and $\max \lambda = 1$, we split R into

$$|R| \leq \iint_D (1 - x)|v|(L(|v| + |v_x|) + M_\Phi) dx dy = \iint_D (R_1 + R_2) dx dy,$$

$$R_1 = (1 - x)Lv^2 + (1 - x)M_\Phi |v|, \quad R_2 = (1 - x)L|v_x||v|.$$

Using Young's inequality with $\varepsilon > 0$: $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ we have

$$(1 - x)M_\Phi |v| = M_\Phi [(1 - x)|v|] \leq \varepsilon(1 - x)^2 v^2 + \frac{1}{4\varepsilon} M_\Phi^2.$$

For $x \in [0, 1]$, $0 \leq 1 - x \leq 1$, so $(1 - x)^2 \leq 1 - x$.

Consequently, $(1 - x)M_\Phi |v| \leq \varepsilon(1 - x)v^2 + \frac{1}{4\varepsilon} M_\Phi^2$,

$$R_1 \leq (1 - x)Lv^2 + \varepsilon(1 - x)v^2 + \frac{1}{4\varepsilon} M_\Phi^2.$$

For $R_2 = (1 - x)L|v_x||v| = (\sqrt{1 - x}|v_x|)(L\sqrt{1 - x}|v|) \leq \varepsilon(\sqrt{1 - x}|v_x|)^2 + \frac{1}{4\varepsilon}(L\sqrt{1 - x}|v|)^2 = \varepsilon(1 - x)v_x^2 + \frac{L}{4\varepsilon}(1 - x)v^2$.

Let in $R_2 \quad \varepsilon = \frac{3}{4}$, then we obtain $R_2 \leq \frac{3}{4}(1-x)v_x^2 + \frac{L}{3}(1-x)v^2$.

$$|R| \leq \iint_D \left[(1-x) \left(\varepsilon + \frac{4L}{3} \right) v^2 + \frac{3}{4}(1-x)v_x^2 + \frac{1}{4\varepsilon} M_\Phi^2 \right] dx dy.$$

Let $C = \varepsilon + \frac{4L}{3}$, $C_1 = \frac{1}{4\varepsilon} M_\Phi^2$,

$$|R| \leq C \iint_D (1-x)v^2 dx dy + \frac{3}{4}(1-x)v_x^2 dx dy + C_1.$$

Define the energy function for $y \in [0, 1]$ $E(y) = \int_0^1 (1-x)v^2 dx$.

Integrating over the domain $D = (0, 1) \times (0, y)$, the reduced energy inequality becomes

$$\frac{1}{2}E(y) + \frac{3}{4} \iint_D (1-x)v_x^2 dx d\eta \leq C \int_0^y E(\eta) d\eta + C_1.$$

Dropping the non-negative gradient term $\frac{3}{4} \iint_D (1-x)v_x^2 dx d\eta$, we obtain the integral inequality

$$E(y) \leq 2C \int_0^y E(\eta) d\eta + 2C_1.$$

Applying Grönwall's inequality yields a uniform bound on $E(y)$ for all $y \in [0, 1]$

$$E(y) \leq 2C_1 e^{2C} \leq 2C_1 e^{2C} = M_1.$$

Substituting this back into the gradient term gives

$$\iint_D (1-x)v_x^2 dx d\eta \leq M_2.$$

Since $v(0, y) = 0$, for any $x \in [0, 1]$ $v(x, y) = \int_0^x v_\xi(\xi, y) d\xi$.

Using Cauchy-Schwarz estimates, we obtain

$$v^2 \leq \left(\int_0^x 1 d\xi \right) \left(\int_0^x v_\xi^2(\xi, y) d\xi \right) \leq \int_0^1 v_\xi^2(\xi, y) d\xi.$$

Now we multiply the equation

$$v_y - v_{xxx}(x, y) = \lambda f(x, y, v + \Phi, v_x + \Phi_x) - \Phi_y = \lambda F(x, y, v, v_x) \quad (13)$$

by v_y and integrate over $D_y = (0, 1) \times (0, y)$

$$\int_0^y \int_0^1 v_y^2 dx d\eta - \int_0^y \int_0^1 v_{xxx} v_y dx d\eta = \lambda \int_0^y \int_0^1 F v_y dx d\eta.$$

Integration by parts for the third-order spatial term yields

$$- \int_0^1 v_{xxx} v_y dx = -v_{xx} v_y \Big|_{x=0}^{x=1} + \frac{1}{2} \frac{d}{dy} \int_0^1 v_{xx}^2 dx.$$

Boundary terms vanish or are sign-definite depending on the boundary behavior ($v(0, y) = 0 \Rightarrow v_y(0, y) = 0$, $v(1, y) = 0 \Rightarrow v_y(1, y) = 0$, $v_x(0, y) = 0 \Rightarrow v_{xy}(0, y) = 0$).

Hence

$$\int_0^y ||v_y||_{L_2(0,1)}^2 d\eta + \frac{1}{2} ||v_{xx}||_{L_2(0,1)}^2 - \frac{1}{2} ||v_{xx}(\cdot, 0)||_{L_2(0,1)}^2 = \lambda \int_0^y \int_0^1 F v_y dx d\eta.$$

Let $z = v_y$. Differentiating the equation with respect to y , we have

$$y - z_{xxx}(x, y) = \lambda \frac{\partial}{\partial y} f(x, y, v + \Phi, v_x + \Phi_x) - \Phi_{yy},$$

$$\Phi(x, y) = \phi_1(y) + x\phi_1(y) + x^2(\psi_1(y) - \phi_1(y) - \phi_2(y)).$$

Applying the multiplier $2z$ and integrating by x from 0 to 1 , we obtain

$$\int_0^1 z z_y dx - \int_0^1 z z_{xxx} dx = \lambda \int_0^1 \frac{\partial F}{\partial y} z dx. \quad \int_0^1 z z_{xxx} dx = z z_{xx} \Big|_{x=0}^{x=1} - \frac{1}{2} z_x^2 \Big|_{x=0}^{x=1}.$$

Since $z(0, y) = z_x(0, y) = z(1, y) = 0$, the boundary evaluations vanish except at $x=1$:

$$\int_0^1 z z_{xxx} dx = -\frac{1}{2} z_x^2(1, y) \leq 0.$$

Hence

$$\frac{1}{2} \frac{d}{dy} |z(\cdot, y)|_{L_2(0,1)}^2 + \frac{1}{2} z_x^2(1, y) \leq \int_0^1 \left| \frac{\partial F}{\partial y} \right| |z| dx.$$

$$\int_0^1 \left| \frac{\partial F}{\partial y} \right| |z| dx \leq \left\| \frac{\partial F}{\partial y} \right\|_{L_2(0,1)} ||z||_{L_2(0,1)} \leq \frac{1}{2\varepsilon} \left\| \frac{\partial F}{\partial y} \right\|_{L_2(0,1)}^2 + \frac{\varepsilon}{2} ||z||_{L_2(0,1)}^2,$$

$$F = \lambda f(x, y, v + \Phi, v_x + \Phi_x) - \Phi_y.$$

Assuming standard Lipschitz conditions on $f(x, y, v + \Phi, v_x + \Phi_x)$

$$F \leq C(1 + |v| + |v_x|).$$

Squaring both sides and integrating over $x \in (0, 1)$

$$||F||_{L_2(0,1)}^2 \leq C_1(1 + ||v||_{L_2(0,1)}^2 + ||v_x||_{L_2(0,1)}^2).$$

Hence,

$$\frac{1}{2} \frac{d}{dy} ||z(\cdot, y)||_{L_2(0,1)}^2 + \frac{1}{2} z_x^2(1, y) \leq \frac{1}{2\varepsilon} \left\| \frac{\partial F}{\partial y} \right\|_{L_2(0,1)}^2 + \frac{\varepsilon}{2} ||z||_{L_2(0,1)}^2.$$

Collecting the inequalities over the domain $D = (0, 1) \times (0, 1)$, we obtain

$$||v_x(\cdot, y)||_{L_2(0,1)}^2 \leq C_1 \int_0^y ||v_y(\cdot, \eta)||_{L_2(0,1)}^2 d\eta + C_2.$$

$$\text{By Gronwall's lemma } ||v_y(\cdot, y)||_{L_2(0,1)}^2 \leq C_2 e^{C_1 y} \leq C_2 e^{C_1} = M.$$

Integrating with respect to y over $[0, 1]$ gives the desired uniform bound

$$||v_y(\cdot, y)||_{L_2(0,1)}^2 \leq M,$$

where $M = M(||\phi_1||, ||\phi_2||, ||\psi_1||, ||f||)$ is a constant completely independent of $\lambda \in [0, 1]$.

Since $(1-x)$ is bounded away from zero locally, the uniform bound on $\int_0^1 \int_0^1 v^2_x dx dy$ along with boundary conditions $v(x, 0) = 0$ and $v(0, y) = 0$ yields a

uniform bound in W_2^1 .

By embedding, we obtain the uniform estimate

$$\|v\|_{C(\bar{D})} \leq M.$$

$v(x, y) = u(x, y) - \Phi(x, y)$ or $u(x, y) = v(x, y) + \Phi(x, y)$, this directly yields the desired uniform a priori estimate

$$\|u\|_C \leq M.$$

DISCUSSION

The results obtained in this study address a key problem in the qualitative theory of differential equations: establishing criteria for well-posedness, global solvability, and uniqueness of solutions for linear and weakly nonlinear equations of odd order with multiple characteristics. By combining energy integral estimates, methods of reduction to integral operators, and embedding techniques in metric spaces within the framework of the parameter continuation approach, this work resolves the primary research problem formulated in the introduction.

The theoretical analysis convincingly demonstrates that asymmetric boundary conditions of the Cattabriga type are not merely sufficient but fundamentally necessary for the well-posed formulation of initial-boundary value problems for systems of higher odd order. The study's primary novelty lies in the proof of the existence and uniqueness of a global solution—addressing issues linked to non-self-adjointness and the asymmetry of wave propagation directions—which allows for effective regularization through the use of function spaces with an appropriate structure.

CONCLUSION

This study addresses the issues of well-posedness, existence, and global uniqueness of solutions to initial-boundary value problems (IBVPs) for linear and weakly nonlinear differential equations of odd order with multiple characteristics. By combining energy integral estimates, the reduction of the problem to integral operators, embeddings into complete metric spaces, and the parameter continuation method, a comprehensive functional-analytic framework was developed; this framework enabled the resolution of difficulties associated with non-self-adjoint operators and the asymmetry of wave propagation directions.

The main theoretical results and contributions of the work are as follows:
Necessity of asymmetric boundary data: it was proven that asymmetric boundary conditions are not merely sufficient but structurally necessary to ensure the well-posedness of IBVPs for higher-order odd-order equations with multiple characteristics.
Uniqueness and energy estimates: based on the establishment of definite-sign identities for energy integrals, precise a priori estimates were derived, proving the strict uniqueness of classical solutions and ruling out the possibility of energy growth at spatial boundaries.

Global existence via the parameter continuation method: transforming the differential problem into an equivalent system of integral operators within a metric space allowed for the effective application of the parameter continuation method. This facilitated the proof of global solvability over arbitrary finite intervals.

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