DEGENERATE CASE IN INVESTIGATING CRITICAL POINTS IN TWO-VARIABLE ECONOMIC OPTIMIZATION PROBLEMS

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Abstract. *In economics, most of the optimization problems reduce to finding critical points of an objective function. For a two-variable differential function $f: \mathbb{R}^2 \to \mathbb{R}$, a critical point where both partial derivatives vanish can be classified according to the discriminant*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) \neq 0,$$

where the point may be a local maximum, a local minimum, or a saddle point. However, if D=0, the second-derivative test is inconclusive. We present accessible methods such as Taylor expansion and Splitting methods. The proposed approaches are accessible to undergraduate students and researchers encountering degenerate critical points in optimization and mathematical modelling.

Keywords: *Polynomial, partial derivative, determinant, critical point, two-variable function, Taylor series.*

I. INTRODUCTION

Let $f(x, y)$ be a two-variable differentiable function. The point (x_0, y_0) is called a critical point of f if both partial derivatives vanish at this point, i.e.,

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0$$

Consider the Hessian matrix associated with f:

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \quad (1)$$

A critical point can be a local maximum, a local minimum, or a saddle point of the function depending on the determinant of the Hessian,

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0). \quad (2)$$

It is well known (see e.g., Refs. [1-2] for the details) that

- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum;
- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum;
- If $D < 0$, then (x_0, y_0) is a saddle point;

Thus, if $D \neq 0$, we can fully classify the critical point, which is then called *non-degenerate*. For example, let

$$f(x, y) = x^2 + y^2.$$

Then, its partial derivatives are

$$f_x = 2x, \qquad f_y = 2y.$$

So the unique critical point is (0,0). Here

$$D = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 4 > 0$$

and

$$f_{xx}(0,0) = 2.$$

Hence $(0,0)$ is a local minimum.

However, in many optimization problems, we encounter cases where the Hessian matrix is not invertible, i.e., $D = 0$. In such cases, the critical point is called *degenerate*. Standard textbooks often recommend examining higher-order derivatives or using graphical methods. In this paper, we investigate the degenerate case and discuss various approaches with examples.

Classical differential methods are often inadequate for solving optimization problems involving degeneracy. To address these difficulties, more advanced frameworks have been developed within modern optimization theory. Variational analysis extends classical calculus in a systematic way, enabling the study of optimization problems under irregular conditions [3-5]. Within this framework, critical points are characterized using generalized derivatives and higher-order optimality conditions that do not rely on the non singularity of the Hessian. Further progress in nonlinear programming has led to stronger second-order optimality conditions that remain valid even when the Hessian matrix is singular [6]. These results have been complemented by directional and variational second-order approaches, which provide a more refined analysis of stationary points in degenerate settings [7,8].

Let (x_0, y_0) be a critical point of $f(x, y)$. Then

$$g(x, y) = f(x + x_0, y + y_0)$$

has a critical point at (0,0). Thus, investigating an arbitrary critical point reduces to studying the origin, (0,0), for a related function. Hence, without loss of generality, we assume throughout the paper that the critical point is at (0,0).

Before proceeding, we recall some important definitions and results.

II. CLASSIFICATION OF DEGENERATE CRITICAL POINTS.

Taylor series of two-variable functions. Let $f(x, y)$ be an analytic function around (0,0). Then, it can be expanded in convergent Taylor series (see, [9]) as

$$f(x, y) = f(0,0) + P_1(x, y) + \frac{1}{2!}P_2(x, y) + \frac{1}{3!}P_3(x, y) + \cdots$$

where,

$$P_1(x, y) = f_x(0,0)x + f_y(0,0)y,$$
$$P_2(x, y) = f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2,$$

and for $P_k(x, y)$, we have

$$P_k(x, y) = \sum_{i=0}^{k} \frac{\partial^k f}{\partial x^i \partial y^{k-i}}(0,0)\, x^i y^{k-i}.$$

Each $P_k(x, y)$ is a homogeneous polynomial of degree k, i.e.,

$$P_k(tx, ty) = t^k P_k(x, y).$$

If (0,0) is a critical point, the linear part vanishes and we obtain

$$f(x,y) = f(0,0) + \frac{1}{2}\left[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right] + \cdots$$

The Hessian matrix H in Eq. (1) is a 2×2 matrix and its rank r can 2, 1, or 0. If $r = 2$, then both eigenvalues of H are non-zero and the critical point is non-degenerate. Depending on r, we can classify the function's behaviour near (0,0) as follows.

Generalized splitting lemma (Morse) lemma ([10-12]). Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a smooth (analytic) function with a non-degenerate critical point at (0,0). Then there exist neighbourhoods U and V of the origin and a smooth change of variables

$$\varphi: U \to V$$

such that

$$(p,q) = \varphi(x,y)$$

with

$$(0,0) = \varphi(0,0)$$

and the following cases hold:

Case $r = 2$ (non-degenerate). The function f can be written as

$$f(p,q) = f(0,0) + \lambda_1 p^2 + \lambda_2 q^2$$

where λ_1 and λ_2 are the eigenvalues of H. Thus, near the origin, the function behaves as a quadratic form. Since both eigenvalues are non-zero, we may scale coordinates so that $\lambda_1, \lambda_2 = \pm 1$.

Consequently,

- If both eigenvalues are positive,

$$f(p,q) = f(0,0) + p^2 + q^2$$

and (0,0) is a local minimum;

- If both are negative,

$$f(p,q) = f(0,0) - p^2 - q^2$$

and (0,0) is a local maximum;

- If the eigenvalues have opposite signs,

$$f(p,q) = f(0,0) + p^2 - q^2$$

and (0,0) is a saddle point.

Case $r = 1$ (degenerate). The function can be written as

$$f(p,q) = f(0,0) + \lambda p^2 + g(q)$$

Where $\lambda = \pm 1$ is the unique non-zero eigenvalue of H, and g is a smooth function of one variable with

$$g(0) = g'(0) = g''(0) = 0.$$

Hence, its Taylor expansion contains no constant, linear, or quadratic terms:

$$g(x) = a_k x^k + a_{k+1} x^{k+1} + \cdots, \qquad k \geq 3.$$

Three subcases arise:

a) If k is odd, (0,0) is a saddle point, regardless of the sign of λ.

b) If k is even and $a_k > 0$. Then, (0,0) is
   - a saddle point if $\lambda = -1$;
   - a minimum if $\lambda = 1$.

c) If k is even and $a_k < 0$. Then, (0,0) is

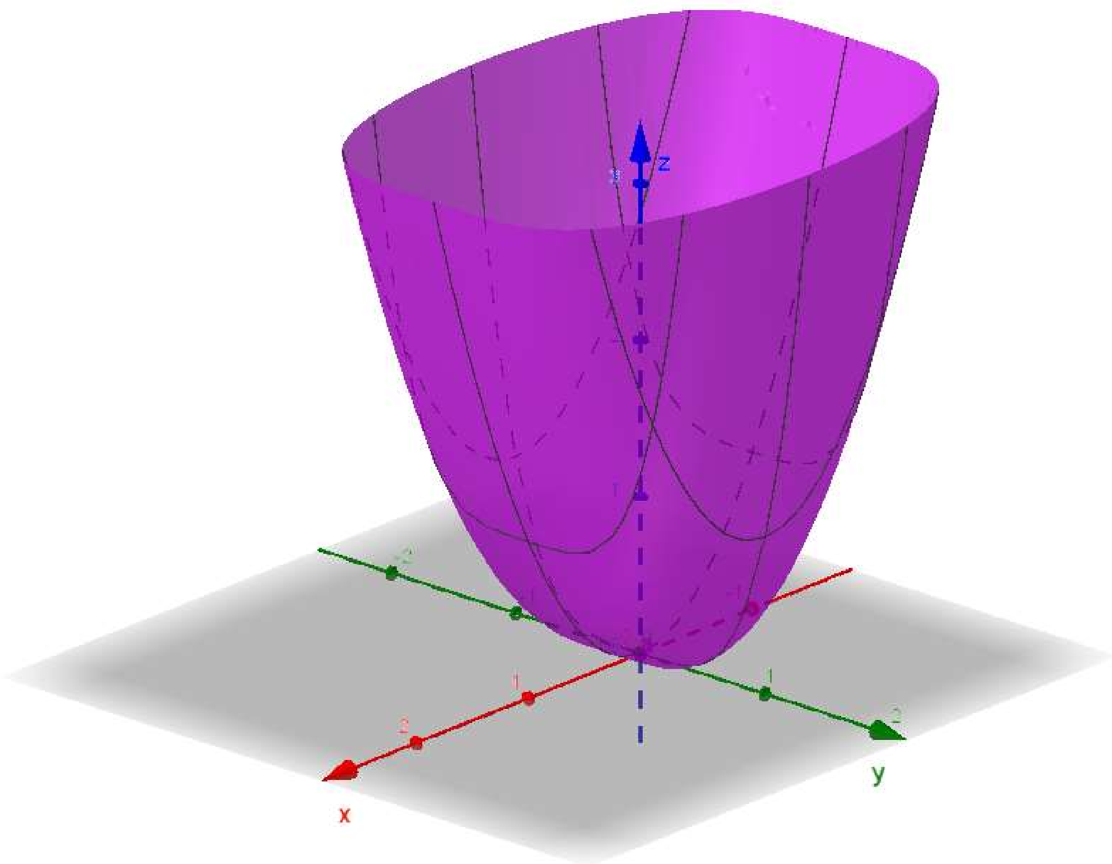
- a maximum if $\lambda = -1$;
- a saddle point if $\lambda = 1$.

Examples:

- (0,0) is a saddle point for the functions

$$f(x,y) = x^2 + y^3$$
$$f(x,y) = x^2 - y^3$$
$$f(x,y) = x^2 - y^4$$
$$f(x,y) = -x^2 + y^4$$

- $f(x,y) = -x^2 - y^4$: (0,0) is a local maximum.
- $f(x,y) = x^2 + y^4$, then (0,0) is a minimum (see Fig. 1).



- ***Figure 1****. Illustration of* $f(x,y) = x^2 + y^4$

We note that these classifications remain valid in the presence of higher-order terms, as the lowest-degree non-zero terms dominate locally.

Case $r = 0$ (degenerate). The function can be written as

$$f(p,q) = f(0,0) + h(p,q),$$

where h is a smooth function of two variables of at least third order. Then

$$h(p,q) = H_k(x,y) + (higher\ terms),$$

where H_k is a homogeneous polynomial of order k (see, e.g., Ref. [13-14]). To classify the critical point, it suffices to study the behaviour of H_k.

Thus, classifying critical points reduces to investigating homogeneous polynomials of degree $k \geq 3$.

Let

$$H_k(x,y) = a_0x^k + a_1x^{k-1}y + a_2x^{k-2}y^2 + \cdots + a_ky^k$$

be a homogeneous polynomial of degree k.

Several methods are available:

1. Polar coordinates: Set

$$x = r\cos\theta, y = r\sin\theta.$$

Then

$$H_k(x,y) = r^k\left(a_0\sin^k\theta + a_1\sin^{k-1}\theta\cos\theta + \cdots + a_k\cos^k\theta\right) = r^k\mathcal{H}(\theta).$$

The sign of $\mathcal{H}(\theta)$ determines the local behaviour.

Example 1. $f(x,y) = x^4 - 6x^2y^2 + y^4$.

Using polar coordinates, we obtain

$$f(x,y) = r^4(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta).$$

Simplify:

$$\mathcal{H}(\theta) = (\sin^2\theta + \cos^2\theta)^2 - 8\cos^2\theta\sin^2\theta = 1 - 2\sin^2(2\theta).$$

Since $1 - 2\sin^2(2\theta)$ changes from -1 to 1, e $\mathcal{H}(\theta)$ also changes sign, therefore (0,0) is a saddle point.

Example 2. $f(x,y) = -x^6 - 7x^4y^2 + x^2y^4 - y^6$.

In polar coordinates,

$$f(x,y) = -r^6(\cos^6\theta + 7\cos^4\theta\sin^2\theta - \cos^2\theta\sin^4\theta + \sin^6\theta)$$

$$\mathcal{H}(\theta) = \cos^6\theta + 7\cos^4\theta\sin^2\theta - \cos^2\theta\sin^4\theta + \sin^6\theta$$

Next, we simplify the $\mathcal{H}(\theta)$ as follows:

$$\mathcal{H}(\theta) = (\cos^2\theta + \sin^2\theta)^3 + 4\cos^4\theta\sin^2\theta - 4\cos^2\theta\sin^4\theta$$
$$= 1 + 4\cos^2\theta\sin^2\theta\,(\cos^2\theta - \sin^2\theta) = 1 + \sin^2 2\theta\cos 2\theta$$

The expression $1+\sin^2 2\theta\cos 2\theta$ is always strictly positive, as the term $\sin^2 2\theta\cos 2\theta$ cannot reach -1. Hence, $\mathcal{H}(\theta) > 0$, so $f(x,y) < f(0,0)$ for all $(x,y) \neq (0,0)$. Thus, (0,0) is a local maximum.

2. Factoring out x^k:

$$H_k(x,y) = x^k\left(a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_k\left(\frac{y}{x}\right)^k\right)$$

Let $t = \frac{y}{x}$. Then, $H_k(x,y) = x^k\mathcal{H}(t)$. For example, let

$$H_4(x,y) = x^4 - x^2y^2 + y^4 = x^4(t^4 - t^2 + 1)$$

The quadratic function $s^2 - s + 1, s = t^2$ has a leading term of 1 and its discriminant is negative, so it is always positive.

3. Test along axes: If

$$H_k(x,0) = ax^k, H_k(0,y) = by^k \text{ with } ab < 0,$$

then (0,0) is a saddle point.

For example, consider

$$f(x,y) = x^4 + 3x^2y^2 - 2x^3y - 2y^4.$$

Then,

$$f(x,0) = x^4 \text{ and } f(0,y) = -2y^4.$$

The product of the coefficients is $-2 < 0$, so (0,0) is a saddle point.

For odd k, if $a \neq 0$ or $b \neq 0$, the critical point is also a saddle point.

Example. Let

$$f(x,y) = x^3 - 3xy^2.$$

Here,

$$f(x,0) = x^3 \ (a = 1 \neq 0),$$

so (0,0) is a saddle point.

III. APPLICATION: AN ECONOMICS PROBLEM

Consider the following problem:

A firm produces a single output using two inputs, x and y. The firm's total revenue is

$$R(x,y) = 10\ln(x+y),$$

and total cost is

$$C(x,y) = 2x + 2y + (x-2)^6.$$

Find the maximum profit.

The profit function is

$$P(x,y) = R(x,y) - C(x,y)$$

Thus,

$$P(x,y) = 10\ln(x+y) - 2x - 2y - (x-2)^6$$

Critical points. The partial derivatives are

$$\begin{cases} P_x = \dfrac{10}{x+y} - 2 - 6(x-2)^5 = 0 \\ P_y = \dfrac{10}{x+y} - 2 = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ x + y = 5 \end{cases}$$

So, (2,3) is the unique critical point of $P(x,y)$.

Second derivative test: Compute the second partial derivatives at (2,3):

$$P_{xx} = \frac{-10}{(x+y)^2} - 30(x-2)^4 \Rightarrow P_{xx}(2,3) = -\frac{2}{5},$$

$$P_{yy} = \frac{-10}{(x+y)^2} \Rightarrow P_{yy}(2,3) = -\frac{2}{5},$$

$$P_{xy} = \frac{-10}{(x+y)^2} \Rightarrow P_{xy}(2,3) = -\frac{2}{5}.$$

Then,

$$D = f_{xx}f_{yy} - f_{xy}^2 = 0,$$

so the second derivative test is inconclusive.

Taylor expansion method. We investigate P near the point (2,3) using the change of variables $x = u + 2$, $y = v + 3$, which shift the critical point to the origin in the uv-plane.

Then,

$$P(x,y) = 10\ln(x+y) - 2x - 2y - (x-2)^6$$

becomes

$$\Phi(u,v) = 10\ln(u+v+5) - 2u - 2v - 10 - u^6.$$

Expand $\ln(u+v+5)$ around $u + v = 0$:

$$\ln(u+v+5) \approx \ln 5 + \frac{(u+v)}{5} - \frac{(u+v)^2}{50} + \frac{(u+v)^3}{375} + O((u+v)^4).$$

Substituting into $\Phi(u,v)$:

$$\Phi(u,v) = 10\ln 5 + 2(u+v) - \frac{(u+v)^2}{5} + \frac{2(u+v)^3}{75} - 2(u+v) - 10 - u^6 + O((u+v)^4).$$

For sufficiently small u and v, the dominant terms are

$$\Phi(u,v) \approx 10\ln 5 - 10 - \frac{(u+v)^2}{5} - u^6$$

since the higher-order terms including the cubic terms can be locally neglected.

The expression

$$-\frac{(u+v)^2}{5} - u^6$$

is zero only when $u = 0$ and $u + v = 0$, i.e., at the origin $(0,0)$, and negative elsewhere. Hence, $(0,0)$ is a strict local maximum of $\Phi(u,v)$.

Consequently, $(2,3)$ is a strict maximum of $\mathrm{P}(\mathrm{x},\mathrm{y})$, and the maximum profit is

$$P(2,3) = 10(\ln 5 - 1).$$

IV. CONCLUSION

In this work, we studied the classification of critical points when the second-derivative test is inconclusive. Degenerate critical points occur when the Hessian determinant equals zero, which limits the usefulness of the standard second-derivative test for functions of several variables. In economic applications, these cases occur quite often. They commonly appear in equilibrium analysis, utility maximization, and cost minimization problems where the curvature is flat or locally ambiguous.

When $D = 0$, second-order information alone is insufficient to determine the local behavior of the function. Modern optimization theory addresses this issue by employing higher-order Taylor expansions, directional derivatives, and variational analysis. These approaches enable a more detailed classification of stationary points by capturing subtle variations in curvature beyond the quadratic level.

Recent research shows that degenerate critical points often carry meaningful economic content, such as threshold equilibria or unstable decision boundaries. Thus, rather than being exceptions, they offer valuable insights into the structure of economic models. The techniques discussed here, especially higher-order Taylor expansions and splitting methods, are both accessible for undergraduate instruction and relevant to advanced work in variational analysis and nonlinear optimization. We illustrated some of them here with examples including an economics application.

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