BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER EQUATION WITH MULTIPLE CHARACTERISTICS

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Abstract. *In this article the author studied one boundary value problem for a third-order nonlinear equation with multiple characteristics. The unique solvability to the problem was proven. The uniqueness of the solution to the boundary value problem was proven by the method of energy. To prove the existence of a solution to this problem, an auxiliary problem was considered, for which the Green function was constructed. By solving an auxiliary problem, the original problem was reduced to a integral equation. The solvability of the integral equation was established using the contraction mapping principle.*

Keywords*: Boundary value problem, uniqueness, existence, integral equations, the contraction mapping principle.*

INTRODUCTION

The study of nonlinear partial differential equations of odd order remains a challenging and comparatively less explored area of modern mathematical analysis. In particular, we consider the equation

$$L(u) \equiv u_{xxx} - u_y = f(x, y), \quad (MC)$$

which belongs to the class of equations with multiple characteristics. Equations of this type are characterized by the degeneracy of their characteristic structure, leading to substantial analytical difficulties in the investigation of existence, uniqueness, and regularity of solutions. The presence of multiple characteristics significantly complicates the application of classical methods developed for strictly hyperbolic or elliptic equations.

Equations with multiple characteristics (MC) arise naturally in various problems of physics and mechanics. They appear in models describing wave propagation, dispersive media, and certain anisotropic processes where higher-order spatial effects play a crucial role. Owing to their mixed-type behavior and nonstandard characteristic geometry, such equations are of considerable theoretical interest and possess important applied significance. Nevertheless, many fundamental questions concerning boundary value problems, well-posedness, and qualitative properties of solutions remain insufficiently studied.

A prominent representative of nonlinear dispersive equations of odd order is the well-known Korteweg-de Vries equation (KdV),

$$u_y + uu_x + \beta u_{xxx} = 0 \qquad (KdV)$$

which has been extensively investigated over the past decades. The (KdV) equation occupies a central position in the theory of nonlinear wave propagation in weakly dispersive media. It serves as a fundamental model in fluid dynamics, plasma physics, and nonlinear optics, describing the evolution of long waves of small amplitude. The rich mathematical structure of the (KdV) equation, including its integrability, soliton solutions, and infinite hierarchy of conservation laws, has stimulated the development of powerful analytical methods.

In comparison with the (KdV) equation, linear and nonlinear equations with multiple characteristics of the form (MC) have received considerably less attention. Their analytical treatment requires refined functional-analytic techniques and careful examination of the associated characteristic manifolds. In particular, the interplay between dispersion, anisotropy, and nonhomogeneous terms f(x,y) gives rise to new phenomena that are not present in classical dispersive models.

This paper aims to further investigate equation (MC), focusing on the formulation of appropriate boundary value problems and the analysis of their solvability. We are studying the qualitative behavior of solutions and establishing results that contribute to the general theory of higher-order equations with multiple characteristics. Our approach combines methods of modern partial differential equations, functional analysis, and the theory of distributions, providing new insights into this important but insufficiently explored class of problems.

LITERATURE REVIEW

The KdV equation finds applications in diverse fields, such as fluid dynamics (e.g., modeling gravitational waves in shallow water and nonlinear Rossby waves), plasma physics (e.g., describing ion-acoustic waves), electrical engineering (e.g., analyzing nonlinear circuits), and even epidemiology (e.g., simulating the time evolution of infected individuals during an epidemic), etc. [2 – 4].

In connection with these and other practical applications, the study of boundary value problems for odd-order equations with multiple characteristics is relevant.

Note that some linear boundary value problems for a equation with multiple characteristics of an odd order were considered in [1, 7-11].

METHODOLOGY

The uniqueness of the solution to the boundary value problem was proven by the method of energy.

To prove the existence of a solution to this problem, an auxiliary problem was considered, for which the Green function was constructed. By solving an auxiliary problem, the original problem was reduced to a integral equation. The solvability of the integral equation was established using the contraction mapping principle.

STATEMENT OF THE PROBLEM

Problem A. It is required to determine in domain $D=\{(x,y):0<x<1, 0<y<1\}$ function $u(x,y)$ that has the following properties:

1) $u(x,y)\in C^{3,1}_{x,y}(D)\cap C^{2,0}_{x,y}(\overline{D})$;that has the following properties

2) which is a regular solution to the following equation:

$$u_{xxx}-u_y=u \qquad (1)$$

in domain D;

3) satisfying the following conditions

$$u(x,0)=u_0(x), \quad 0\le x\le 1, \qquad (2)$$

$$u_{xx}(0,y)=\varphi_1(y), 0\le y\le 1, \qquad (3)$$

$$u_x(0,y)=\varphi(y), 0\le y\le 1, \qquad (4)$$

$$u_{xx}(1,y)=\psi(y), 0\le y\le 1, \qquad (5)$$

and matching conditions

$$\varphi_1(0)=u_0''(0),\ \varphi_2(0)=u_0'(0),\ \psi(0)=u_0''(1).$$

ANALYSIS AND RESULTS

UNIQUENESS OF THE SOLUTION

Theorem (Uniqueness of solution).

The solution to problem A is unique.

Proof. Let there be two solutions to the considered problem, u_1 and u_2. Consider their difference $w = u_1 - u_2$. With regard to w, we obtain the following problem:

$$w_{xxx}-w_y=w \qquad (1_0)$$

$$w(x,0)=0,\ 0\le x\le 1, \qquad (2_0)$$

$$w_{xx}(0,y)=0, 0\le y\le 1, \qquad (3_0)$$

$$w_x(0,y)=0,\ 0\le y\le 1, \qquad (4_0)$$

$$w_{xx}(1,y)=0, 0\le y\le 1, \qquad (5_0)$$

Let us prove that $w(x,y)\equiv 0$.

Having integrated the following identity

$$w(w_{xxx}-w_y)=w^2$$

over domain D, taking into account boundary conditions (2_0) - (5_0), we introduce the following notation:

$$\int_0^1\left(ww_{xx}-\frac{1}{2}w_x^2\right)\Bigg|_{x=0}^{x=1}dy-\frac{1}{2}\int_0^1 w^2\Big|_{y=0}^{y=1}dy=\iint_D w^2dxdy,$$

$$\frac{1}{2}\int_0^1\left(w_x^2\right)\Big|^{x=1}dy+\frac{1}{2}\int_0^1 w^2\Big|_{y=1}dy+\iint_D w^2dxdy=0. \qquad (6)$$

Then from (11) we obtain the following conditions:

$$w_x(1,y)=0,\ 0\le y\le 1,\ w(x,1)=0, 0\le x\le 1,\ w(x,y)=0,(x,y)\in D.$$

So $w(x,y)=0,\ (x,y)\in D.$

EXISTENCE OF THE SOLUTION

Before proceeding to the proof of the existence of a solution to Problem A, it is necessary to study the following auxiliary problem.

Problem B. It is required to determine in domain D a regular solution

$$u(x,y)\in C^{2n+1,1}_{x,y}(D)\cap C^{2n,0}_{x,y}(\overline{D})$$

to equation

$$L(u)\equiv u_{xxx}-u_y=f(x,y) \qquad (1_1)$$

satisfying the following conditions (2) - (5).

Let us construct the Green function for problem B.

The following identity holds:

$$zL(q)-qM(z)=\frac{\partial}{\partial\xi}\left\{\sum_{k=0}^{2}(-1)^k\frac{\partial^k z}{\partial\xi^k}\frac{\partial^{2n-k}q}{\partial\xi^{2n-k}}\right\}-\frac{\partial}{\partial\eta}(zq), \qquad (6)$$

where $M\equiv-\frac{\partial^3}{\partial\xi^3}+\frac{\partial}{\partial\eta}$ is the operator conjugate to operator $L\equiv\frac{\partial^3}{\partial\xi^3}-\frac{\partial}{\partial\eta}$,

and $z(\xi,\eta)$ and $q(\xi,\eta)$ are rather smooth functions.

Integrating identity (13) over domain D, we obtain

$$\iint_D(zL(q)-qM(z))d\xi d\eta=\int_\Gamma\left(\sum_{k=0}^{2}(-1)^k\frac{\partial^k z}{\partial\xi^k}\frac{\partial^{2n-k}q}{\partial\xi^{2n-k}}\right)d\eta+(zq)d\xi. \qquad (7)$$

Now, in formula (14), for functions *q* and *z,* we take functions *u (x, y)* - any regular solution to equation $Lu=f(x,y)$ and $U(x,y;\xi,\eta)$ - fundamental solution to equation $Lu=0$ ([1]).

As ε tends to zero and considering the following equality ([1])

$$\lim_{\varepsilon\to0}\int_0^1U(x,y;\xi,y-\varepsilon)u(\xi,y-\varepsilon)d\xi=\pi u(x,y),$$

we obtain

$$\pi u(x,y)=\int_0^y\sum_{k=0}^{2}(-1)^k\frac{\partial^k U}{\partial\xi^k}\frac{\partial^{2n-k}u}{\partial\xi^{2n-k}}\Bigg|_{\xi=0}^{\xi=1}d\eta+\int_0^1U(x,y;\xi,0)u(\xi,0)d\xi-\iint_D U(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta. \ (8)$$

Let now $W(x,y;\xi,\eta)$ be any regular solution to the following equation:

$$-v_{\xi\xi\xi}+v_\eta=0, \qquad (9)$$

and *u* (*x*, *y*) is any regular solution to equation $Lu=f(x,y)$. Now, assuming in formula (8) that $z=W(x,y;\xi,\eta)$, $q=u$, we have

$$\iint_D W(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta=\int_0^y\sum_{k=0}^{2}(-1)^k\frac{\partial^k W}{\partial\xi^k}\frac{\partial^{2n-k}u}{\partial\xi^{2n-k}}\Bigg|_{\xi=0}^{\xi=1}d\eta-\int_0^1(Wu)\Big|_{\eta=0}^{\eta=y}d\xi. \ (10)$$

From equalities (7) and (10), we find

$$\pi u(x,y)=\int_0^y \sum_{k=0}^{2}(-1)^k \frac{\partial^k (U-W)}{\partial \xi^k}\frac{\partial^{2n-k}u}{\partial \xi^{2n-k}}\Bigg|_{\xi=0}^{\xi=1} d\eta+\int_0^1 W(x,y;\xi,y)u(\xi,y)d\xi+$$

$$+\int_0^1 (U(x,y;\xi,0)-W(x,y;\xi,0))u(\xi,0)d\xi-\iint_D (U(x,y;\xi,\eta)-W(x,y;\xi,\eta))f(\xi,\eta)d\xi d\eta.$$

(11)

If regular solution $W(x,y;\xi,\eta)$ to equation (9) satisfies the following conditions:

$$W(x,y;\xi,\eta)\big|_{\eta=y}=0,\ W_{\xi\xi}\big|_{\xi=0}=U_{\xi\xi}\big|_{\xi=0},\ W_{\xi}\big|_{\xi=1}=U_{\xi}\big|_{\xi=1},\ W_{\xi\xi}\big|_{\xi=1}=U_{\xi\xi}\big|_{\xi=1}. \quad (12)$$

then from formula (11) we obtain

$$u(x,y)=-\frac{1}{\pi}\int_0^y Gu_{\xi\xi}\big|_{\xi=0}d\eta+\frac{1}{\pi}\int_0^y G_{\xi}u_{\xi}\big|_{\xi=0}d\eta+\frac{1}{\pi}\int_0^1 Gu\big|_{\eta=0}d\xi+$$
$$+\frac{1}{\pi}\int_0^y Gu_{\xi\xi}\big|_{\xi=1}d\eta-\frac{1}{\pi}\iint_D G(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta, \quad (13)$$

where $G(x,y;\xi,\eta)=U(x,y;\xi,\eta)-W(x,y;\xi,\eta)$ we call the Green function of problem B.

Formula (13) gives a solution to problem B.

Note that for function $G(x,y;\xi,\eta)$ the same estimates as for function $U(x,y;\xi,\eta)$ are true ([1]).

$$\left.\begin{array}{l} U(x.y;\xi,\eta)=\begin{cases}(y-\eta)^{-\frac{1}{3}}f\left(\dfrac{x-\xi}{(y-\eta)^{\frac{1}{3}}}\right), & x\neq\xi,\ y>\eta,\\ 0, & y\le\eta.\end{cases} \\ V(x.y;\xi,\eta)=\begin{cases}(y-\eta)^{-\frac{1}{3}}\varphi\left(\dfrac{x-\xi}{(y-\eta)^{\frac{1}{3}}}\right), & x>\xi,\ y>\eta,\\ 0, & y\le\eta.\end{cases}\end{array}\right\} \quad (14)$$

Let's define the following functions:

$$f(t)=\int_0^{+\infty}\cos(\lambda^3-\lambda t)d\lambda,\ -\infty<t<+\infty,$$

$$\varphi(t)=\int_0^{\infty}\left[\exp\left(-\lambda^3-\lambda t\right)+\sin\left(\lambda^3-\lambda t\right)\right]d\lambda,\quad t=\frac{x-\xi}{(y-\eta)^{1/3}}.$$

Functions $f(t)$, $\varphi(t)$ called Airy functions, satisfy the following equation (see [1]):

$$z''(t)+\frac{1}{3}tz(t)=0. \qquad (15)$$

The following relations are true for functions $U(x,y;\xi,\eta)$, $V(x,y;\xi,\eta)$:

$$|U(x,y;\xi,\eta)| < \frac{C}{(y-\eta)^{1/3}}$$

$$\left.\begin{array}{l}\left|\dfrac{\partial^{i+j}U(x,y;\xi,\eta)}{\partial x^i \partial y^j}\right| \\ \left|\dfrac{\partial^{i+j}V(x,y;\xi,\eta)}{\partial x^i \partial y^j}\right|\end{array}\right\} < C_1 \frac{|x-\xi|^{\frac{2i+6j-1}{4}}}{|y-\eta|^{\frac{2i+6j-1}{4}}}, \qquad (16)$$

as $\dfrac{x-\xi}{(y-\eta)^{1/3}} \to +\infty,\ \ i+j \geq 1,\ \ C>0,\ \ C_1>0,$

$$\left|\frac{\partial^{i+j}U(x,y;\xi,\eta)}{\partial x^i \partial y^j}\right| < \frac{C_2}{|y-\eta|^{\frac{i+3j+1}{4}}} \exp\left(-C_3 \frac{|x-\xi|^{\frac{3}{2}}}{|y-\eta|^{\frac{1}{2}}}\right), \ (17)$$

as $\dfrac{x-\xi}{(y-\eta)^{1/3}} \to -\infty,\ \ i+j \geq 1,\ \ C_2>0,\ \ C_3>0.$

Solution to problem (9), (12) is sought in the following form:

$$W(x,y;\xi,\eta) = \int_\eta^y U(1,\tau;\xi,\eta)\alpha_1(x,y,\tau)d\tau + $$
$$+\int_\eta^y V(1,\tau;\xi,\eta)\alpha_2(x,y,\tau)d\tau + \int_\eta^y U(0,\tau;\xi,\eta)\beta(x,y,\tau)d\tau . \qquad (18)$$

The equation (18) has a solution belonging to class $\alpha_1 \epsilon C^1(0,y), \alpha_2 \epsilon C(0,y)$,

$\beta \epsilon C^1(0,y)$.

Theorem. Let, along with the conditions of the uniqueness theorem, the following conditions be satisfied:

$$u_0(x) \in C^3[0;1],\ \varphi_1(y) \in C[0,1],\ \varphi_2(y) \in C^1[0,1], \psi(y) \in C[0,1].$$

Then the solution to problem (1) - (5) exists.

Proof. From (13), we find

$$u(x,y) = -\frac{1}{\pi}\iint_D G(x,y;\xi,\eta)u(\xi,\eta)d\xi d\eta + H(x,y), \qquad (19)$$

$$H(x,y) = -\frac{1}{\pi}\int_0^y G(x,y;0,\eta)\varphi_1(\eta)d\eta + \frac{1}{\pi}\int_0^y G_\xi(x,y;0,\eta)\varphi_2(\eta)d\eta + \frac{1}{\pi}\int_0^y G(x,y;1,\eta)\psi_2(\nu(\eta),\eta)d\eta -$$
$$-\frac{1}{\pi}\int_0^y G(x,y;\xi,0)u_0(\xi)d\xi.$$

(19) is a linear integral equations with respect to $u(x,y)$.

We will prove the unique solvability of this system using the contraction mapping principle.

Let G_θ be a set of functions $F=\{u(x,y)\}$, that are continuous in domain $D_\theta = \{(x,y); 0<x<1,\ 0\leq y\leq \theta\}$ and have on the interval $0\leq y\leq\theta$ bounded norm $\|F\| = \|u\|$, where $\|u\| = \max_{(x,y)\in D_\theta} |u|$.

Let $G_{\theta,N}$ denote subset $\{F : F \in G_\theta, \|F\| \le N\}$ of set G_θ.

Denoting the right-hand side of (19) by $A(u)$, we define the mapping of $A = A(u)$.

Let us show that for some θ and N>0 , for $0 \le y \le \theta$, operator A transforms $G_{\theta,N}$ into itself. That is, inequalities $\|A\| \le K$, are true for $u \in G_{\theta,N}$. For this, we assume that $A(u)$, are defined in $G_{\theta,N}$.

From relation (19), we obtain:

$$|u| \le \frac{3C}{\pi}\left(\|u\| + \|\varphi_1\| + \frac{8}{9}\|\varphi_2\|\theta^{\frac{1}{12}} + \|\psi\| + \|u_0\|\right)\theta^{\frac{2}{3}}.$$

For K we take $K = \dfrac{3C}{2\pi}$, and θ is chosen in such a way that the following inequality is met:

$$\left(\|u\| + \|\varphi_1\| + \frac{8}{9}\|\varphi_2\|\theta^{\frac{1}{12}} + \|\psi\| + \|u_0\|\right)\theta^{\frac{2}{3}} \le 1.$$

Then relation $\|A\| \le K$. holds.

Therefore, operator A maps the set $G_{\theta,N}$ into itself.

Let us now show that with an appropriate choice of θ, operator A is contractive. We have

$$\left|A(u) - A_1(u^*)\right| \le \frac{1}{\pi}\iint_D |u(\xi,\eta) - u^*(\xi,\eta)| G(x,y;\xi,\eta) d\xi d\eta \le \frac{3C}{2\pi}\|u - u^*\|\theta^{\frac{2}{3}}.$$

We choose θ so that inequality $\frac{3C}{2\pi}\theta^{\frac{2}{3}} < 1$ is met.

Then we have:

$$|A(u) - A(u^*)| < \|u - u^*\|.$$

For $\theta < \left[\dfrac{2\pi}{3C}\right]^{\frac{3}{2}}$ operator $A(u)$ is a contraction mapping. Then, by virtue of the principle of contraction mappings, it has a single fixed point $u \in G_{\theta,K}$. We assume that θ is chosen so as to ensure the compressibility of operator $A(u)$, and that operator $A(u)$ maps $G_{\theta,N}$ into itself.

Therefore, u is a solution to system (19) for $0 \le y \le \theta$.

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