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ON A BOUNDARY VALUE PROBLEM FOR AN ODD-ORDER EQUATION WTH MULTIPLE CHARACTERISTICS

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Abstract- In this article the author studied one boundary value problem for a third-order nonlinear equation with multiple characteristics. The unique solvability to the problem was proven. The uniqueness of the solution to the boundary value problem was proven by the method of energy. To prove the existence of a solution to this problem, an auxiliary problem was considered. By solving an auxiliary problem, the original problem was reduced to a system of Hammerstein integral equations.The solvability of a nonlinear system is established by the method of contraction mappings.

Keywords- nonlinearity, uniqueness, existence, system of Hammerstein equations.

I. INTRODUCTION

The following equation refers to poorly studied odd-order equations:

$$L(u)\equiv u_{xxx}-u_y=f(x,y,u,u_x,u_{xx}),\quad (MC)$$

which is called an equation with multiple characteristics (*MC*)([1]).

The well-known Korteweg-de Vries equation *(KdV)*

$$u_y+uu_x+\beta u_{xxx}=0 \qquad (KdV)$$

which is the object of research by many authors and occupies an important place in the study of nonlinear wave propagation in weakly dispersive media [2 – 4].

The Korteweg–de Vries (*KdV*) equation describes the evolution of weakly nonlinear long-wavelength excitations in a medium with dispersion in the high-frequency domain. The KdV equation finds applications in diverse fields, such as fluid dynamics (e.g., modeling gravitational waves in shallow water and nonlinear Rossby waves), plasma physics (e.g., describing ion-acoustic waves), electrical engineering (e.g., analyzing nonlinear circuits), and even epidemiology (e.g., simulating the time evolution of infected individuals during an epidemic), etc. [2 – 6].

I. LITERATURE ANALYSIS

In connection with above and other practical applications, the study of boundary value problems for odd-order equations with multiple characteristics is relevant.

Note that some boundary value problems for a equation with multiple characteristics of an odd order were considered in [1,7 – 9]. This paper employs the method of contraction mappings to investigate a linear boundary value problem for an odd-order nonlinear equation with multiple characteristics.

III. ANALYSIS AND RESULTS
STATEMENT OF THE PROBLEM

PROBLEM A. It is required to determine in domain $D=\{(x,y):0<x<1,0<y\le 1\}$ function $u(x,y)$ that has the following properties:

1) $u(x,y)\in C^{3,1}_{x,y}(D)\cap C^{2,0}_{x,y}(\overline{D})$;

2) which is a regular solution in domain D to the following equation:

$$L(u)\equiv u_{xxx}-u_y=f(x,y,u,u_x) \qquad (1)$$

3) satisfying the following conditions

$$u(x,0)=0,\quad 0\le x\le 1, \qquad (2)$$

$$u_{xx}(0,y)=\varphi_1(y),\quad 0\le y\le 1, \qquad (3)$$

$$u_x(0,y)=\varphi_2(y),\quad 0\le y\le 1, \qquad (4)$$

$$u(1,y)=\psi(y),\quad 0\le y\le 1, \qquad (5)$$

and matching conditions $\varphi_1(0)=\varphi_2(0)=\psi(0)=0$.

UNIQUENESS OF THE SOLUTION

THEOREM (*the uniqueness of the solution*). Let $f(x,y,u,u_x)$ be continuous function of their arguments and for any $|u|<\infty$, $|u_x|<\infty$ satisfying the following condition

$$|f(x,y,u_1,u_{1x})-f(x,y,u_2,u_{2x})|\le L\{|u_1-u_2|+|u_{1x}-u_{2x}|\}, \qquad (6)$$

$$\frac{1}{3}< L \le 1, \qquad (7)$$

where L - some constant.

Then the solution to Problem A is unique.

Proof. Let there be two solutions to the considered problem, u_1 and u_2. Consider their difference $w = u_1 - u_2$. With regard to w, we obtain the following problem:

$$\mathrm{L}(w) \equiv w_{xxx}-w_y=f(x,y,u_1,u_{1x})-f(x,y,u_2,u_{2x}), \qquad (1_0)$$

$$w(x,0)=0,\quad 0\le x\le 1, \qquad (2_0)$$

$$w_{xx}(0,y)=0,\quad 0\le y\le 1, \qquad (3_0)$$

$$w_x(0,y)=0,\quad 0\le y\le 1, \qquad (4_0)$$

$$w(1,y)=0,\quad 0\le y\le 1, \qquad (5_0)$$

Let us prove that $w(x,y)\equiv 0$.

Having integrated the following identity

$$v\,w\,\mathrm{L}(w)\, vwL(w)\equiv vw(w_{xxx}-w_y)=vw(f(x,y,u_1,u_{1x})-f(x,y,u_2,u_{2x})) \qquad (8)$$

over domain D, where $v=\exp(-x-2y)$, taking into account boundary conditions (2_0) - (5_0), we introduce the following notation:

$$I=\frac{1}{2}\int_0^1 vw_x^2\Big|_{x=1}dy+\frac{1}{2}\int_0^1 vw^2\Big|_{y=1}dx+\frac{3}{2}\iint_D vw_x^2dxdy\geq 0. \qquad (9)$$

According to notation (9), alternatively we have

$$I=-\frac{1}{2}\int_0^1 v_{xx}w^2\Big|_{x=0}dy+\frac{1}{2}\iint_D\left(v_y-v_{xxx}\right)w^2dxdy-\iint_D\left(f\left(x,y,u_1,u_{1x}\right)-f\left(x,y,u_2,u_{2x}\right)\right)wvdxdy.$$

In accordance with condition (6) we obtain the following

$$I\leq\frac{1}{2}\int_0^1(L-1)vw^2\Big|_{x=0}dy+\frac{1}{2}\iint_D(1-3L)w^2vdxdy.$$

When condition (7) are satisfied, we arrive at the following inequality

$$0\leq\frac{1}{2}\int_0^1 vw_x^2\Big|_{x=1}dy+\frac{1}{2}\int_0^1 vw^2\Big|_{y=1}dx+\frac{3}{2}\iint_D vw_x^2dxdy+\frac{1}{2}\iint_D(3L-1)vw^2dxdy+\frac{1}{2}\int_0^1(1-L)vw^2\Big|_{x=0}dy\leq 0.$$

Hence,

$$\frac{1}{2}\int_0^1 vw_x^2\Big|_{x=1}dy+\frac{1}{2}\int_0^1 vw^2\Big|_{y=1}dx+\frac{3}{2}\iint_D vw_x^2dxdy+\frac{1}{2}\iint_D(3L-1)vw^2dxdy+\frac{1}{2}\int_0^1(1-L)vw^2\Big|_{x=0}dy=0,$$

from that we obtain the following conditions:

$$\begin{cases} w_x(1,y)=0\ , \ 0\leq y\leq 1,\\ w(x,1)=0, 0\leq x\leq 1,\\ w_x(x,y)=0, (x,y)\in D,\\ w(x,y)=0, (x,y)\in D.\end{cases}$$

So $w(x,y)=0,\ (x,y)\in D.$

EXISTENCE OF THE SOLUTION

Before proceeding to the proof of the existence of a solution to Problem A, it is necessary to study the following auxiliary problem.

PROBLEM B. It is required to determine in domain D a regular solution

$$u(x,y)\in C_{x,y}^{2n+1,1}(D)\cap C_{x,y}^{2n,0}(\overline{D})$$

to equation

$$L(u)\equiv u_{xxx}-u_y=0 \qquad (1_0)$$

satisfying the following conditions

$$u(x,0)=0,\quad 0\leq x\leq 1, \qquad (2)$$

$$u_{xx}(0,y)=\varphi_1(y),\quad 0\leq y\leq 1, \qquad (3)$$

$$u_x(0,y)=\varphi_2(y),\quad 0\leq y\leq 1, \qquad (4)$$

$$u(1,y)=\psi(y),\quad 0\leq y\leq 1, \qquad (5)$$

We seek the solution to problem B in the following form:

$$u(x,y)=\int_0^y U_{xx}(x,y;1,\eta)\alpha_1(\eta)d\eta+\int_0^y V(x,y;0,\eta)\alpha_2(\eta)d\tau+\int_0^y U(x,\tau;0,\eta)\beta(\eta)d\eta\ .$$

Obviously, function $u(x,y)$ satisfies condition $u(x,y)|_{y=0}=0$.

Satisfying the boundary conditions we obtain

$$\int_0^y U_{xxxx}(0,y;1,\eta)\alpha_1(\eta)d\eta+\int_0^y V_{xx}(x,y;0,\eta)|_{x=0}\alpha_2(\eta)d\tau+\int_0^y U_{xx}(x,y;0,\eta)|_{x=0}\beta(\eta)d\eta=\varphi_1(y),\quad(10)$$

$$\int_0^y U_{xxx}(0,y;1,\eta)\alpha_1(\eta)d\eta+\int_0^y V_x(0,y;0,\eta)\alpha_2(\eta)d\tau+\int_0^y U_x(0,y;0,\eta)\beta(\eta)d\eta=\varphi_2(y),\quad(11)$$

$$\int_0^y U_{xx}(x,y;1,\eta)|_{x=1}\alpha_1(\eta)d\eta+\int_0^y V(1,y;0,\eta)\alpha_2(\eta)d\tau+\int_0^y U(1,y;0,\eta)\beta(\eta)d\eta=\psi(y).\quad(12)$$

Note that for functions $U(x,y;\xi,\eta)$ and $V(x,y;\xi,\eta)$ are true ([1])

$$\left.\begin{aligned}U(x.y;\xi,\eta)&=\begin{cases}(y-\eta)^{-\frac{1}{3}}f\left(\dfrac{x-\xi}{(y-\eta)^{\frac{1}{3}}}\right), & x\neq\xi,\ y>\eta,\\ 0, & y\leq\eta.\end{cases}\\ V(x.y;\xi,\eta)&=\begin{cases}(y-\eta)^{-\frac{1}{3}}\varphi\left(\dfrac{x-\xi}{(y-\eta)^{\frac{1}{3}}}\right), & x>\xi,\ y>\eta,\\ 0, & y\leq\eta.\end{cases}\end{aligned}\right\}\quad(13)$$

$$f(t)=\int_0^{+\infty}\cos(\lambda^3-\lambda t)d\lambda,\ -\infty<t<+\infty,\ \varphi(t)=\int_0^{\infty}\left[\exp(-\lambda^3-\lambda t)+\sin(\lambda^3-\lambda t)\right]d\lambda,\ t=\frac{x-\xi}{(y-\eta)^{1/3}}.$$

Functions $f(t), \varphi(t)$ called Airy functions, satisfy the following equation ([1]):

$$z''(t)+\frac{1}{3}tz(t)=0.\quad(14)$$

The following relations hold:

$$\int_0^{+\infty}f(t)dt=\frac{2}{3}\pi,\ \int_0^{+\infty}\varphi(t)dt=0,\ \int_{-\infty}^{0}f(t)dt=\frac{\pi}{3}.\quad(15)$$

$$f(0)=\frac{\sqrt{3}}{2}\int_0^{+\infty}e^{-t^3}dt,\quad f'(0)=\frac{\sqrt{3}}{2}\int_0^{+\infty}t\cdot e^{-t^3}dt$$

$$\varphi(0)=\frac{\sqrt{3}}{2}\int_0^{+\infty}e^{-t^3}dt,\quad \varphi'(0)=\frac{\sqrt{3}}{2}\int_0^{+\infty}t\cdot e^{-t^3}dt.$$

The following relations are true for functions $U(x,y;\xi,\eta), V(x,y;\xi,\eta)$:

$$|U(x,y;\xi,\eta)|<\frac{C}{(y-\eta)^{1/3}}$$

$$\left.\begin{aligned}&\left|\frac{\partial^{i+j}U(x,y;\xi,\eta)}{\partial x^i\partial y^j}\right|\\ &\left|\frac{\partial^{i+j}V(x,y;\xi,\eta)}{\partial x^i\partial y^j}\right|\end{aligned}\right\}<C_1\frac{|x-\xi|^{\frac{2i+6j-1}{4}}}{|y-\eta|^{\frac{2i+6j-1}{4}}},\quad(16)$$

as $\dfrac{x-\xi}{(y-\eta)^{1/3}} \to +\infty, \;\; i+j \ge 1, \;\; C>0, \;\; C_1>0,$

$$\left|\frac{\partial^{i+j}U(x,y;\xi,\eta)}{\partial x^i \partial y^j}\right| < \frac{C_2}{|y-\eta|^{\frac{i+3j+1}{4}}} \exp\left(-C_3\frac{|x-\xi|^{\frac{3}{2}}}{|y-\eta|^{\frac{1}{2}}}\right), \qquad (17)$$

as $\dfrac{x-\xi}{(y-\eta)^{1/3}} \to -\infty, \;\; i+j \ge 1, \;\; C_2>0, \;\; C_3>0.$

(10) - (12) is a system of integral equations of the Volterra type for unknown functions $\alpha_k(y)$, $k=1,2$, and $\beta(y)$.

The solution to the system (10)-(12) can be represented as

$$\alpha_1(y) = \frac{3}{\pi}\int_0^y R(y,\tau)\psi(\tau)d\tau + \int_0^y K_{11}(y,s)\varphi_2'(s)ds + \int_0^y K_{12}(y,s)\varphi_1(\eta)d\eta + \frac{3}{\pi}\psi(y), \qquad (18)$$

where R(y,τ) is the resolvent kernel of the equation

$$\alpha_1(y) = \int_0^y K(y,\tau)a_1(\tau)d\tau + F(y),$$

$$K(y,\tau) = \frac{3\sqrt{3}}{2\pi^2\varphi'(0)}\int_\tau^y \frac{1}{(y-\eta)^{\frac{1}{3}}}\varphi\left(\frac{1}{(y-\eta)^{\frac{1}{3}}}\right)\left[\int_\tau^\eta \frac{1}{(z-\tau)^{\frac{1}{3}}}\frac{\partial}{\partial z}\left(\frac{1}{(\eta-z)^{\frac{1}{3}}} f\left(\frac{-1}{(\eta-z)^{\frac{1}{3}}}\right)\right)dz\right]d\eta +$$

$$+\frac{9f'(0)}{2\pi^2\varphi'(0)}\int_\tau^y \frac{1}{(y-\eta)^{\frac{1}{3}}}\varphi\left(\frac{1}{(y-\eta)^{\frac{1}{3}}}\right)\frac{1}{(\eta-\tau)^{\frac{5}{3}}} f^{\prime\prime\prime\prime}\left(\frac{-1}{(\eta-\tau)^{\frac{1}{3}}}\right)d\eta +$$

$$+\frac{9f'(0)}{2\pi^2}\int_\tau^y \frac{1}{(y-\eta)^{\frac{1}{3}}}\varphi\left(\frac{1}{(y-\eta)^{\frac{1}{3}}}\right)\frac{1}{(\eta-\tau)^{\frac{5}{3}}} f^{\prime\prime\prime\prime}\left(\frac{-1}{(\eta-\tau)^{\frac{1}{3}}}\right)d\eta,$$

$$F(\tau) = \frac{3}{\pi}\psi(\tau) - \frac{3\sqrt{3}}{2\pi^2\varphi'(0)}\int_0^\tau \varphi_2'(s)ds\int_s^\tau \frac{V(1,\tau;0,\eta)}{(\eta-s)^{\frac{1}{3}}}d\eta - \frac{9f'(0)}{2\pi^2\varphi'(0)}\int_0^\tau V(1,\tau;0,\eta)\varphi_1(\eta)d\eta +$$

$$+\frac{9}{2\pi^2}\int_0^\tau U(1,\tau;0,\eta)\varphi_1(\eta)d\eta,$$

$$K_{11}(y,s) = -\frac{3\sqrt{3}}{2\pi^2\varphi'(0)}\int_s^y R(y,\tau)\left[\int_s^\tau \frac{V(1,\tau;0,\eta)}{(\eta-s)^{\frac{1}{3}}}d\eta\right]d\tau + \int_s^y \frac{V(1,y;0,\eta)}{(\eta-s)^{\frac{1}{3}}}d\eta,$$

$$K_{12}(y,s) = -\frac{9f'(0)}{2\pi^2\varphi'(0)}\left[V(1,y;0,\eta) - \frac{\varphi'(0)}{f'(0)}U(1,y;0,\eta) + \int_\eta^y R(y,\tau)\left(U(1,\tau;0,\eta)+V(1,\tau;0,\eta)\right)d\tau\right]$$

$$\alpha_2(y) = \int_0^y K_{21}(y,\tau)\varphi_1(s)ds + \int_0^y K_{22}(y,\tau)\varphi_2'(s)ds + \int_0^y K_{23}(y,\tau)\psi(\tau)d\tau - \frac{3f'(0)}{2\pi\varphi'(0)}\varphi_1(y). \qquad (19)$$

$$K_{21}(y,\tau)=-\frac{9f'(0)}{2\pi^2\varphi'(0)}\int_s^y K_2(y,\eta)\left[V(1,\eta;0,s)-\frac{\varphi'(0)}{f'(0)}U(1,\eta;0,s)+\int_s^\eta R(\eta,\tau)(U(1,\tau;0,s)+V(1,\tau;0,s))d\tau\right]d\eta,$$

$$K_{22}(y,\tau)=\frac{\sqrt{3}}{2\pi\varphi'(0)}\frac{1}{(y-s)^{\frac{1}{3}}}-\frac{3\sqrt{3}}{2\pi^2\varphi'(0)}\left[\int_s^y K_2(y,\eta)R(\eta,\tau)\left[\int_s^\tau \frac{V(1,\tau;0,\eta)}{(\eta-s)^{\frac{1}{3}}}d\eta\right]d\tau+\int_s^\eta \frac{V(1,\eta;0,h)}{(h-s)^{\frac{1}{3}}}dh\right]d\eta,$$

$$K_{23}(y,\tau)=\frac{3}{\pi}K_2(y,\tau)-\frac{3}{\pi}\int_\tau^y K_2(y,\eta)R(\eta,\tau)d\eta.$$

$$\beta(y)=\int_0^y K_{31}(y,\tau)\psi(\tau)d\tau+\int_0^y K_{32}(y,s)\varphi_2'(s)ds+\int_0^y K_{33}(y,s)\varphi_1(s)ds-\frac{3}{2\pi}\varphi_1(y).(20)$$

$$K_{31}(y,\tau)=\frac{9}{4\pi^2}U_{xxxx}(0,y;1,\tau)+\frac{3}{2\pi}\int_\tau^y U_{xxxx}(0,y;1,\eta)R(\eta,\tau)d\eta,$$

$$K_{32}(y,s)=-\frac{9\sqrt{3}}{4\pi^3\varphi'(0)}\left[\int_s^y U_{xxxx}(0,y;1,\eta)\left[\int_s^\eta \frac{V(1,\eta;0,h)}{(h-s)^{\frac{1}{3}}}dh+\int_s^\eta R(\eta,\tau)\left[\int_s^\tau \frac{V(1,\tau;0,h)}{(h-s)^{\frac{1}{3}}}dh\right]d\tau\right]d\eta\right],$$

$$K_{33}(y,s)=-$$

$$-\int_s^y U_{xxxx}(0,y;1,\eta)\left[\left(\frac{27f'(0)}{4\pi^3\varphi'(0)}\left(V(1,\eta;0,s)d\eta+\int_s^\eta R(\eta,\tau)V(1,\tau;0,s)d\tau\right)\right)-\frac{27}{4\pi^3}\left(U(1,\eta;0,s)+\int_s^\eta R(y,\tau)U(1,\tau;0,s)d\tau\right)\right]d\eta.$$

From solution the problem B we get

$$u(x,y)=\int_0^y K_{41}(y,s)\varphi_1(s)ds+\int_0^y K_{42}(y,s)\varphi_2'(s)ds+\int_0^y K_{43}(y,s)\psi(s)ds,$$

where

$$K_{41}(y,s)=\int_s^y U_{xx}(x,y;1,\eta)K_{12}(\eta,s)d\eta+\int_s^y V(x,y;0,\eta)K_{21}(\eta,s)d\eta+$$

$$+\int_s^y U(x,y;0,\eta)K_{33}(\eta,s)d\eta+\frac{3f'(0)}{2\pi\varphi'(0)}V(x,y;0,s)-\frac{3}{2\pi}U(x,y;0,s),$$

$$K_{42}(y,s)=\int_s^y U_{xx}(x,y;1,\eta)K_{11}(\eta,s)d\eta+\int_s^y V(x,y;0,\eta)K_{22}(\eta,s)d\eta+\int_s^y U(x,y;0,\eta)K_{32}(\eta,s)d\eta,$$

$$K_{43}(y,s)=\frac{3}{\pi}\int_s^y U_{xx}(x,y;1,\eta)R(\eta,s)d\eta+\int_s^y V(x,y;0,\eta)K_{23}(\eta,s)d\eta+\int_s^y U(x,y;0,\eta)K_{31}(\eta,s)d\eta.$$

Here $K_{42}(y,s)$, $K_{43}(y,s)$ are continuous functions and $|K_{41}(y,s)|<C\cdot(y-s)^{-1/3}$.

So, if

$$\varphi_1(y) \in L_2(0,1),\ \varphi_2(y) \in C^1(0,1),\ \psi(y) \in C(0,1)$$

then problem B is solvable.

THEOREM(***the existence of the solution***). Let, along with the conditions of the uniqueness theorem, the following conditions be satisfied:

$$\varphi_1(y) \in L_2(0;1), \varphi_2(y) \in C^1(0;1),\ \psi(y) \in C(0;1),$$

$$\frac{\partial}{\partial y}(f(x,y,u,u_x)) \in C(\overline{D}),\ f(x,0,u(x,0),u_x(x,0)) = 0,$$

and $f(x,y,u,u_x) \in C(D)$ for $(x,y) \in D$ and for $|u| < M$

$$|f(x,y,u,u_x)| < M_1,\ |f_y(x,y,u,u_x)| < M_2, |f_u(x,y,u,u_x)| < M_3, |f_{u_x}(x,y,u,u_x)| < M_4.$$

Then the solution to problem (1) - (5) exists.

Proof. Using the solution to problem B, the solution to problem A can be sought in the form

$$u(x,y) = \int_0^y K_{41}(y,s)\varphi_1(s)ds + \int_0^y K_{42}(y,s)\varphi_2'(s)ds \ +$$

$$+\int_0^y K_{43}(y,s)\psi(s)ds \ -\frac{1}{\pi}\iint_D U(x,y;\xi,\eta)f(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta))d\xi d\eta. \quad (21)$$

Differensiating (26) by x we take

$$u_x(x,y) = \int_0^y K_{41x}(y,s)\varphi_1(s)ds + \int_0^y K_{42x}(y,s)\varphi_2'(s)ds \ +$$

$$+\int_0^y K_{43x}(y,s)\psi(s)ds \ -\frac{1}{\pi}\iint_D U_x(x,y;\xi,\eta)f(\xi,\eta,u(\xi,\eta),u_\xi(\xi,\eta))d\xi d\eta. \quad (22)$$

System (21) - (22) is a system of nonlinear Hammerstein integral equations with respect to $u(x,y), u_x(x,y)$.

We will prove the unique solvability of this system using the contraction mapping principle.

Let G_θ be a set of functions $F = \{u(x,y), u_x(x,y)\}$, that are continuous in domain $D_\theta = \{(x,y); 0 < x < 1,\ 0 \le y \le \theta\}$ and have on the interval $0 \le y \le \theta$ bounded norm $\|F\| = \|u\| + \|u_x\|$, where $\|u\| = \max\limits_{(x,y)\in D_\theta} |u|,\ \|u_x\| = \max\limits_{(x,y)\in D_\theta} |u_x|$.

Let $G_{\theta,N}$ denote subset $\{F : F \in G_\theta, \|F\| \le N\}$ of set G_θ.

Denoting the right-hand sides of (21)-(22) by $A_i(u,u_x),\ i = 1,2$ respectively, we define the mapping of $A = (A_1(\circ), A_2(\circ))$.

Let us show that for some θ and $N>0$, for $0 \le y \le \theta$, operator A transforms $G_{\theta,N}$ into itself. That is, inequalities $\|A_i\| \le \frac{K_i}{2}$, $i=1,2$, are true for $(u, ,u_x) \in G_{\theta,N}$. For this, we assume that $A_i(u,u_x)$, $i=1,2$, are defined in G_{θ_i,N_i}, $i=1,2$, respectively.

From relation (21), we obtain:

$$|u| \le \left(\frac{3C}{2}\|\varphi_1\| + \|\varphi_2'\|\theta_1^{\frac{1}{3}} + \|\psi\|\theta_1^{\frac{1}{3}} + \frac{3C}{2\pi}M_1 \right)\theta_1^{\frac{2}{3}}.$$

For K_1 we take $K_1 = 2\left(\frac{3C}{2}\|\varphi_1\| + \|\varphi_2'\|\theta_1^{\frac{1}{3}} + \|\psi\|\theta_1^{\frac{1}{3}} + \frac{3C}{2\pi}M_1 \right)\theta_1^{\frac{2}{3}}.$

and θ_1 is chosen in such a way that the following inequality is met

$$\left(\frac{3C}{2}\|\varphi_1\| + \|\varphi_2'\|\theta_1^{\frac{1}{3}} + \|\psi\|\theta_1^{\frac{1}{3}} + \frac{3C}{2\pi}M_1 \right)\theta_1^{\frac{2}{3}} \le 1.$$

Then relation $\|A_1\| \le \frac{K_1}{2}$ holds.

Anologically we get $\|A_2\| \le \frac{K_2}{2}$.

Assume that $K = \max_j K_j$, $j=1,2$, and due to further reduction in θ_j, $j=1,2$, it is required that the inequality $|A_j| \le \frac{K}{2}$ be met.

Then for $0 \le y \le \theta_j$, $j=1,2$, relation $|A_j| \le \frac{K}{2}$, $j=1,2$ holds.

Therefore, choosing $\theta = \min_j \theta_j$, $j=1,2$ for $0 \le y \le \theta$, operator A maps the set $G_{\theta,N}$ into itself.

Let us now show that with an appropriate choice of θ , operator A is contractive. We have

$$|A_1(u,u_x) - A_1(u^*,u^*_x)| \le \frac{3CL}{2\pi}\left[\|u-u^*\| + \|u_x - u_x^*\|\right]\theta_1^{\frac{2}{3}},$$

We choose θ_1 so that inequality $\frac{3Cl_2}{2\pi}\theta_1^{\frac{2}{3}} < \frac{1}{2}$ is met.

In a similar way we have $|A_2(u,u_x) - A_2(u^*,u^*_x)| \le \frac{4C_1L}{3\pi}\left[\|u\| + \|u_x\|\right]\theta_2^{\frac{3}{4}}.$

We choose θ_2 so that inequality $\frac{4C_1l_1}{3\pi}\theta_2^{\frac{3}{4}} < \frac{1}{2}$ is met.

Then, we have

$$|A_1(u,u_x) - A_1(u^*,u^*_x)| < \frac{\|u-u^*\| + \|u_x - u_x^*\|}{2},$$

$$|A_2(u,u_x) - A_2(u^*,u^*_x)| < \frac{\|u-u^*\| + \|u_x - u_x^*\|}{2}.$$

For $\theta = \min_j \theta_j$, $j = 1,2$ operator $A(u, u_x)$ is a contraction mapping. Then, by virtue of the principle of contraction mappings, it has a single fixed point $(u, u_x) \in G_{\theta,K}$. We assume that θ is chosen so as to ensure the compressibility of operator $A(u, u_x)$, and that operator $A(u, u_x)$ maps $G_{\theta,N}$ into itself.

Therefore, (u, u_x) is a solution to system (21) - (22) for $0 \le y \le \theta$.

The calculation of $\theta = \min_j \theta_j$, $j = 1,2$ does not depend on initial data. This independence allows extending the solution along y over the interval [0, 1]. ([16])

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