DOI: 10.5281/zenodo.15651264
Link: https://zenodo.org/records/15651264

THE ESSENTIAL SPECTRUM OF THE SCHRÖDINGER OPERATOR FOR A THREE-PARTICLE SYSTEM WITH MASSES $m_1 = m_2 = \infty$ AND $m_3 < \infty$

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***Abstract.** We consider a family of parameter-dependent discrete Schrödinger operators corresponding to the Hamiltonian of a system of three arbitrary particles (either fermions or bosons) with masses $m_1 = m_2 = \infty$ and $m_3 < \infty$, on the integer lattice, $\mathbb{Z}^3$. The interactions of particles are described via zero-range attractive forces. Using the direct-integral decomposition method, the three-particle problem is reduced to the study of simpler two-particle Schrödinger operators, called the channel (or fiber) operators. Using the channel operators, we find that the essential spectrum consists of a finite union of real segments.*

***Keywords:** Schrödinger operator, spectrum, essential spectrum, Fredholm determinant.*

I. INTRODUCTION

The study of systems consisting of three particles with various masses and interaction energies has long been of particular interest in both physics and mathematics. Three-particle systems exhibit physical phenomena that do not arise in two-particle systems. One such phenomenon is the Efimov effect [1], wherein three identical bosons can form bound states even though the interaction between any two of them is insufficient to form a bound pair. Equivalently, under certain conditions, the corresponding three-body Schrödinger operators may possess infinitely many eigenvalues accumulating at the edges of the essential spectrum. Efimov's theoretical predictions have been experimentally confirmed in ultracold gases of caesium atoms [2]. The existence of the Efimov effect for discrete Schrödinger operatorsвЂ"lattice analogues of continuous Schrödinger operatorsвЂ"has also been established [3, 4, 5].

In recent years, the spectral properties of discrete Schrödinger operators on lattices have been extensively investigated for various one-, two-, and three-particle systems (see, e.g., [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]). In Ref. [6], three-particle Schrödinger operators describing a system of two light particles and one heavy particle on the three-dimensional lattice $\mathbb{Z}^3$ were studied. Descriptions and estimates for the essential spectrum were obtained.

In this paper, we study a Hamiltonian of the form $H = H_0 - V$, where H_0 is a non-perturbative operator and V is a potential function, corresponding to a system of two heavy (infinite-mass) and one light (finite-mass) particles on $\mathbb{Z}^3$. The particles interact via parameter-dependent short-range pair potentials. This model operator is significant in solid-state physics [3, 4]. The spectral properties of the discrete Schrödinger operator H are analyzed, with particular attention to its discrete spectrum, which exhibits features not present in systems consisting solely of light particles (see, e.g., [21]).

The infiniteness of the discrete spectrum has been demonstrated in the one-dimensional case in [22], and in the two-dimensional case in [23]. Although a similar approach is employed for the three-dimensional case, the problem becomes substantially more challenging due to the convergence properties of the integrals over $\mathbb{T}^3$. This, in turn, allows for the possible existence of threshold resonances of the operator H, in addition to eigenvalues. We establish that, depending on the interaction energies, infinitely many eigenvalues may emerge either from both threshold eigenvalues and threshold resonances, or solely from threshold eigenvalues.

Unlike for the continuous case, the discrete three-particle Hamiltonian operator H is not rotationally invariant. However, using the technique of separation of variables, it can be represented as a direct integral of a family of bounded discrete Schrödinger operators $H(K), K \in \mathbb{T}^d$, called the fiber operators [5, 17, 18]. In physical literature, K is referred to as a quasi-momentum. As a result, the study of the spectrum of the three-particle Schrödinger operators is reduced to the study of the fiber operators $H(K), K \in \mathbb{T}^d$. According to the boundedness of the operator $H(K)$, its essential spectrum, $\sigma_{ess}(H(K))$, consists of a union of at most countably many bounded segments (see [19, 20]). In our work, we study the essential spectrum of parameter-dependent discrete Schrödinger operators corresponding to the Hamiltonian of a system of three arbitrary particles (either fermions or bosons) with masses $m_1 = m_2 = \infty$ and $m_3 < \infty$, on the integer lattice, $\mathbb{Z}^3$.

The plan of the paper is as follows. In section 1, the Hamiltonian of a system of three particles is described in coordinate and momentum spaces as bounded self-adjoint operators. We also give the decomposition of the two-particle and three-particle energy operators into von Neumann direct integrals. In section 2, we present the channel operators, two-particle discrete Schrödinger operators $h_\alpha(k), k \in \mathbb{T}^3$, $\alpha = 1,2,3$, corresponding to the three two-particle subsystems of the three-particle system and their properties. In section 3, the main result describes the essential spectrum of $H(K)$ in terms of the spectra of the channel operators. The last section is conclusion.

Throughout the paper we adopt the following notations: $\mathbb{Z}^3$ is the integer lattice of dimension three, $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3 = (-\pi, \pi]^3$ is the three-dimensional torus (the first Brillouin zone, i.e., the dual group of $\mathbb{Z}^3$) equipped with the Haar measure, the subscripts $\alpha, \beta, \gamma \in \{1,2,3\}$ are pairwise different numbers.

II. Three-particle discrete Schrödinger operator on the lattice $\mathbb{Z}^3$

II.1 Coordinate representation

We first describe the three-particle system for particles of arbitrary mass. In coordinate representation, the total Hamiltonian $\widehat{H}$ associated with a system of three particles moving on the three-dimensional lattice $\mathbb{Z}^3$ and interacting via the short-range pair potentials is defined in $\ell^2((\mathbb{Z}^3)^3)$ as

$$\widehat{H} = \widehat{H}_0 - \widehat{V},$$

where

$$\widehat{H}_0 := \frac{1}{2m_1}\Delta \otimes \hat{I} \otimes \hat{I} + \frac{1}{2m_2}\hat{I} \otimes \Delta \otimes \hat{I} + \frac{1}{2m_3}\hat{I} \otimes \hat{I} \otimes \Delta$$

and

$$\widehat{V} := \widehat{V}_1 + \widehat{V}_2 + \widehat{V}_3.$$

Here m_α is the mass of the particle α, $\alpha = 1,2,3$, $\hat{I}$ is the identity operator in $\ell^2(\mathbb{Z}^3)$, and Δ is the standard discrete Laplacian acting as a multidimensional Laurent-Toeplitz-type operator in $\ell^2(\mathbb{Z}^3)$,

$$\Delta\hat{f}(p) := \sum_{s\in\mathbb{Z}^3} \hat{\varepsilon}(s)\hat{f}(p+s), \quad \hat{f} \in \ell^2(\mathbb{Z}^3),$$

with

$$\hat{\varepsilon}(s) = \begin{cases} 6, & if \quad s = 0 \\ -1, & if \quad |s| = 1, \\ 0, & if \quad |s| > 1 \end{cases} \quad (1.1)$$

and the pair-potential $\hat{V}_\alpha$, which describes the interaction between particles β and γ, is a multiplication operator by the Kronecker delta function of mass μ_α ($\mu_\alpha > 0$),

$$(\hat{V}_\alpha\hat{f})(x_1,x_2,x_3) = \mu_\alpha\delta_{x_\beta,x_\gamma}\hat{f}(x_1,x_2,x_3), \ \hat{f} \in \ell^2((\mathbb{Z}^3)^3), \ (x_1,x_2,x_3) \in (\mathbb{Z}^3)^3.$$

We note that under these conditions, the total Hamiltonian $\hat{H}$ is a bounded self-adjoint operator in $\ell^2((\mathbb{Z}^3)^3)$.

1.2 Momentum representation

The three-particle system in momentum space is represented by the Hamiltonian $H = \mathcal{F}_3\hat{H}\mathcal{F}_3^{-1}$ as

$$H = H_0 - V,$$

with

$$H_0 = \mathcal{F}_3\hat{H}_0\mathcal{F}_3^{-1}, \quad V = \mathcal{F}_3\hat{V}\mathcal{F}_3^{-1},$$

where $\mathcal{F}_3$ is the standard Fourier transform and $\mathcal{F}_3^{-1}$ is its inverse.

Then, the free Hamiltonian H_0 is a multiplication operator

$$(H_0f)(p) = (\varepsilon_1(p_1) + \varepsilon_2(p_2) + \varepsilon_3(p_3))f(p), \ p = (p_1,p_2,p_3) \in (\mathbb{T}^3)^3,$$

where the real-valued continuous function ε_α (α=1,2,3), called *the dispersion relation of the α-th normal mode* associated with the free particle α, is defined as

$$\varepsilon_\alpha(p) = \frac{1}{m_\alpha}\epsilon(p), \ \epsilon(p) = \sum_{j=1}^{3} (1 - \cos p^{(j)}), \ p = (p^{(1)},p^{(2)},p^{(3)}) \in \mathbb{T}^3. \quad (1.2)$$

The perturbation operator V is of the form $V = V_1 + V_2 + V_3$, where

$$(V_\alpha f)(p) = \frac{\mu_\alpha}{(2\pi)^9}\int_{(\mathbb{T}^3)^3} \delta(p_\alpha - q_\alpha)\delta(p_\beta + p_\gamma - q_\beta - q_\gamma)f(q_1,q_2,q_3)\, dq_1dq_2dq_3, \ f \in L^2((\mathbb{T}^3)^3),$$

with $\delta(\cdot)$ being the Dirac delta function.

1.3 Decomposition of $\boldsymbol{H}$ and representations of fiber operators

Let $\{\hat{U}_s\}_{s\in\mathbb{Z}^3}$ be the abelian group of the shift operators on the Hilbert space $\ell^2((\mathbb{Z}^3)^3)$,

$$(\hat{U}_s\hat{f})(x_1,x_2,x_3) = \hat{f}(x_1+s,x_2+s,x_3+s), \ x_1,x_2,x_3,s \in \mathbb{Z}^3.$$

Via the Fourier transform $\mathcal{F}_3$, the operator $\hat{U}_s$, $s \in \mathbb{Z}^3$ is unitarily equivalent to a unitary multiplication operator U_s acting in $L^2((\mathbb{T}^3)^3)$ as

$$(U_sf)(p_1,p_2,p_3) = e^{-\mathrm{i}(s,p_1+p_2+p_3)}f(p_1,p_2,p_3), \ f \in L^2((\mathbb{T}^3)^3).$$

Let $\pi_{\alpha\beta}: (\mathbb{T}^3)^3 \to (\mathbb{T}^3)^2$, $\alpha, \beta \in \{1,2,3\}$ be the projection function

$$\pi_{\alpha,\beta}(p_1, p_2, p_3) = (p_\alpha, p_\beta).$$

For a given $K \in \mathbb{T}^3$, define the subset $\mathbb{F}_K \subset (\mathbb{T}^3)^3$ as

$$\mathbb{F}_K := \{(p_1, p_2, p_3) \in (\mathbb{T}^3)^3:\ p_1 + p_2 + p_3 = K\}.$$

Let $\pi_{\alpha\beta K}$ be the restriction of $\pi_{\alpha\beta}$ onto $\mathbb{F}_K$,

$$\pi_{\alpha\beta K}: \mathbb{F}_K \to (\mathbb{T}^3)^2,\ \alpha, \beta \in \{1,2,3\}.$$

Then, the function $\pi_{\alpha\beta K}$ is a bijective map with the inverse $\pi_{\alpha\beta K}^{-1}: (\mathbb{T}^3)^2 \to \mathbb{F}_K$ defined as

$$(\pi_{\alpha K})^{-1}(p_\alpha, p_\beta) = (p_\alpha, p_\beta, K - p_\alpha - p_\beta).$$

Hence, for any $K \in \mathbb{T}^3$, $\mathbb{F}_K$ is homeomorphic to $(\mathbb{T}^3)^2$.

We have the following decomposition of the space $L^2((\mathbb{T}^3)^3)$ into the direct integral

$$L^2((\mathbb{T}^3)^3) = \int_{K\in\mathbb{T}^3} \oplus L^2(\mathbb{F}_K) dK. \qquad (1.3)$$

Correspondingly, the operator U_s, $s \in \mathbb{Z}^3$ can also be decomposed into the direct integral

$$U_s = \int_{K\in\mathbb{T}^3} \oplus U_s(K) dK,$$

where

$$U_s(K) = e^{-\mathrm{i}(s,K)} I_{L^2(\mathbb{F}_K)},$$

with $I_{L^2(\mathbb{F}_K)}$ being the identity operator on the Hilbert space $L^2(\mathbb{F}_K)$. Obviously, the Hamiltonian H commutes with the operators U_s for any $s \in \mathbb{Z}^3$, hence by [24, Theorem XIII.84] the operator H can be rewritten as the von Neumann direct integral

$$H = \int_{K\in\mathbb{T}^3} \oplus \widetilde{H}(K) dK$$

associated with the decomposition (??).

In the physical literature, the parameter $K \in \mathbb{T}^3$ is called the *three-particle quasi-momentum* and the corresponding operators $\widetilde{H}(K)$, $K \in \mathbb{T}^3$, are called the *fiber operators*. For a given $K \in \mathbb{T}^3$, the fiber operator $\widetilde{H}(K)$ acts in $L^2(\mathbb{F}_K)$ as

$$\widetilde{H}(K) = \widetilde{H}_0(K) - \tilde{V},$$

where

$$\widetilde{H}_0(K) f(p) = (\epsilon_1(p_1) + \epsilon_2(p_2) + \epsilon_3(p_3)) f(p),\ p = (p_1, p_2, p_3) \in \mathbb{F}_K, f \in L^2(\mathbb{F}_K)$$

and

$$\tilde{V} = \tilde{V}_1 + \tilde{V}_1 + \tilde{V}_3$$

with

$$(\tilde{V}_\alpha f)(p_\alpha, p_\beta, p_\gamma) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} f(p_\alpha, t, p_\beta + p_\gamma - t) dt, f \in L^2(\mathbb{F}_K).$$

Using the unitary operator

$$U_{\alpha\beta K}: L^2(\mathbb{F}_K) \to L^2((\mathbb{T}^3)^2),\ U_{\alpha\beta K}(g) = g \circ (\pi_{\alpha\beta K})^{-1},\ \alpha, \beta = \{1,2,3\}, \alpha \neq \beta,$$

where

$$\pi_{\alpha\beta K}: \mathbb{F}_K \to \mathbb{T}^3, \quad (\pi_{\alpha\beta K})(p_\alpha, p_\beta, p_\gamma) = (p_\alpha, p_\beta),$$

we define *the momentum representation* of the fiber operator $\widetilde{H}(K)$ as

$$H(K) = U_{\alpha K}\widetilde{H}(K)U_{\alpha K}^{-1}.$$

The operator $H(K)$, $K \in \mathbb{T}^3$ is of the form

$$H(K) = H_0(K) - V_1 - V_2 - V_3.$$

In the coordinates (p,q), $p = p_\alpha, q = p_\beta$, the operators $H_0(K)$ and V_α are defined on the Hilbert space $L_2((\mathbb{T}^3)^2)$ by

$$(H_0(K)f)(p,q) = E(K;p,q)f(p,q), \quad f \in L_2((\mathbb{T}^3)^2), \quad (1.4)$$

and

$$(V_\alpha f)(p,q) = \frac{\mu_\alpha}{(2\pi)^3}\int_{\mathbb{T}^3} f(p,q)dq, \quad f \in L_2((\mathbb{T}^3)^2), \ \alpha = 1,2,3,$$

(1.5)

respectively, where

$$E(K;p,q) = \varepsilon_\alpha(p) + \varepsilon_\beta(q) + \varepsilon_\gamma(K-p-q).$$

In what follows, any operator unitarily equivalent to $\widetilde{H}(K)$ will be called *the three-particle discrete Schrödinger operator*. We use its various representations according to the convenience.

II. Channel Operator

Consider the operators $H_\alpha(K), K \in \mathbb{T}^3$, $\alpha = 1,2,3$, acting on the Hilbert space $L^2((\mathbb{T}^3)^2)$ as

$$H_\alpha(K) = H_0(K) - V_\alpha,$$

where $H_0(K)$ and V_α are defined in (1.4) and (1.5), respectively. $H_\alpha(K), K \in \mathbb{T}^3$ are sometimes refferred to as *the cluster operators* corresponding to the decomposition $\{\{\alpha\},\{\beta,\gamma\}\}$, $\alpha,\beta,\gamma \in \{1,2,3\}$ [20].

The decomposition of the space $L^2((\mathbb{T}^3)^2)$ into the direct integral

$$L^2((\mathbb{T}^3)^2) = \int_{p\in\mathbb{T}^3} \oplus L^2(\mathbb{T}^3)dp$$

allows decomposing $H_\alpha(K)$ into the direct integral

$$H_\alpha(K) = \int_{p\in\mathbb{T}^3} \oplus (h_\alpha(K-p) + \varepsilon_\alpha(p)\mathrm{I})dp, \qquad (2.1)$$

where $h_\alpha(k)$, $k \in \mathbb{T}^3$ $(\alpha = 1,2,3)$ is the two-particle Schrödinger operator corresponding to the subsystem $\{\beta,\gamma\}$ of the three-particle system defined by

$$h_\alpha(k) = h_\alpha^0(k) - v_\alpha, \ k \in \mathbb{T}^3. \ (2.2)$$

The operators $h_\alpha^0(k)$ and v_α are defined on the Hilbert space $L_2(\mathbb{T}^3)$ as

$$(h_\alpha^0(k)f)(p) = E_k^{(\alpha)}(p)f(p), \ f \in L^2(\mathbb{T}^3),$$

and

$$(v_\alpha f)(p) = \frac{\mu_\alpha}{(2\pi)^3}\int_{\mathbb{T}^3} f(q)dq, \ f \in L^2(\mathbb{T}^3), p \in \mathbb{T}^3,$$

respectively, where

$$E_k^{(\alpha)}(p) = \varepsilon_\beta(p) + \varepsilon_\gamma(k-p), \quad p \in \mathbb{T}^3. \qquad (2.3)$$

2.1 Spectral properties of the two-particle discrete Schrödinger operators when $\boldsymbol{m_1 = m_2 = \infty}$ and $\boldsymbol{0 < m_3 < \infty}$

In this subsection, we study discrete spectrum of the operators $h_\alpha(k)$, $k \in \mathbb{T}^3$.

With $m_1 = m_2 = \infty$ and $0 < m_3 < \infty$ and the representation (1.2), the functions (2.3) can be written as

$$E_k^{(1)}(p) = E_k^{(2)}(p) = \epsilon(k-p)/m_3,\ E_k^{(3)}(p) = 0,\ p \in \mathbb{T}^3.$$

Consequently, since the potentials v_α, $\alpha = 1,2,3$ have a convolution-type property, all three two-particle Schrödinger operators depend on the quasi-momentum $k \in \mathbb{T}^3$,

$$h_1 := h_1(k),\ h_2 := h_2(k),\ h_3 := h_3(k).$$

The operators $h_1(k)$, $h_2(k)$ and $h_3(k)$ act on $f \in L^2(\mathbb{T}^3)$ as

$$h_\alpha(k)f(p) = \varepsilon_3(p)f(p) - (v_\alpha f)(p), \alpha = 1,2, \quad h_3(k)f(p) = -(v_3 f)(p).$$

We note that if $\mu_1 = \mu_2$, then $h_1 = h_2$.

As v_α $(\alpha = 1,2,3)$ is a finite rank operator, according to Weyl theorem, the essential spectrum $\sigma_{\text{ess}}(h_\alpha(k))$ of the operator $h_\alpha(k)$ in (2.2) coincides with the spectrum $\sigma(h_\alpha^0(k))$ of the non-perturbed operator $h_\alpha^0(k)$. More specifically,

$$\sigma_{\text{ess}}(h_\alpha(k)) = [E_{\min}^{(\alpha)}(k), E_{\max}^{(\alpha)}(k)],$$

where

$$E_{\min}^{(\alpha)}(k) \equiv \min_{p\in\mathbb{T}^3} E_k^{(\alpha)}(p), \quad E_{\max}^{(\alpha)}(k) \equiv \max_{p\in\mathbb{T}^3} E_k^{(\alpha)}(p).$$

Therefore, in our case we have

$$\sigma_{\text{ess}}(h_1(k)) = \sigma_{\text{ess}}(h_2(k)) = [0,6/m_3] \quad \textit{and} \quad \sigma_{\text{ess}}(h_3(k)) = \{0\}.$$

The Fredholm determinants associated with the operators $h_\alpha(k)$, $\alpha = 1,2$ are defined as

$$\Delta_\alpha(z) = 1 - \frac{\mu_\alpha}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{ds}{\varepsilon_3(s) - z}, \quad z \in \mathbb{C}\backslash[0,6/m_3], \quad \alpha = 1,2.$$

We note that the eigenvalue equation, $h_\alpha(k)f = (h_\alpha^0(k) - v_\alpha)f = zf$, is equivalent to $f = (h_\alpha^0(k) - z)^{-1} v_\alpha f$. Therefore, the we have the following lemma.

Lemma 2.1 *Let $\alpha = 1,2$. The number $z \in \mathbb{C}\backslash[0,6/m_3]$ is an eigenvalue of $h_\alpha(k)$ if and only if*

$$\Delta_\alpha(z) = 0.$$

The solution $C \in \mathbb{C}$ of the equation $\Delta_\alpha(z) = 0$ and the eigenvalue $f \in L^2(\mathbb{T}^3)$ are connected by the relations

$$C = v_\alpha f \quad \textit{and} \quad f = (h_\alpha^0(k) - z)^{-1} C.$$

Denote

$$\mu_0 = \left(\frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{ds}{\varepsilon_3(s)}\right)^{-1}.$$

Lemma 2.2 *a) Let $\alpha = 1,2$. The operator $h_\alpha(k)$ has a unique simple eigenvalue $z = z_\alpha$, not depending on $k \in \mathbb{T}^3$, if $\mu_\alpha > \mu_0$, and has no eigenvalues if $0 < \mu_\alpha \le \mu_0$.*
b) $h_3(k)$ has a unique simple eigenvalue $z = -\mu_3$.

Proof. a) According to the previous lemma, we consider zeros of $\Delta_\alpha(\cdot)$. The function $\Delta_\alpha(\cdot)$ is monotonic decreasing in $(-\infty,0)$ and $\lim_{z\to-\infty}\Delta_\alpha(z)=1$. Therefore, it is easy to see that it has a simple zero if the limit $\lim_{z\to0-}\Delta_\alpha(z)=1-\mu_\alpha/\mu_0$ is negative, and no zeros otherwise.

b) From the definition, we have

$$h_3(k)f=-v_3f=-\frac{\mu_\alpha}{(2\pi)^3}\int_{\mathbb{T}^3}f(q)dq$$

Therefore, $h_3(k)f=zf$ yields $z=-\mu_3$.

The next lemma is a summary of the results of this section.

Lemma 2.3 *For* $\alpha=1,2$, *we have*

$$\sigma(h_\alpha(k))=\begin{cases}[0,6/m_3], & if \quad 0<\mu_\alpha\le\mu_0,\\ \{z_\alpha\}\cup[0,6/m_3], & if \quad \mu_\alpha>\mu_0,\end{cases}$$

where $z_\alpha\in(-\infty,0)$, and

$$\sigma(h_3(k))=\{-\mu_3\}\cup\{0\}.$$

2.2 The spectrum of $\boldsymbol{H_\alpha(K)}$

The direct integral representation (2.1) of the operator $H_\alpha(K)$ yields (see Refs. [21]) the relation

$$\sigma(H_\alpha(K))=\sigma_{\text{two}}(H_\alpha(K))\cup\sigma_{\text{three}}(H_\alpha(K)), \qquad (2.4)$$

where

$$\sigma_{\text{two}}(H_\alpha(K))=\bigcup_{p\in\mathbb{T}^3}\{\sigma_{\text{disc}}(h_\alpha(K-p)))+\varepsilon_\alpha(p)\},\quad \sigma_{\text{three}}(H_\alpha(K))=\bigcup_{p\in\mathbb{T}^3}\{\sigma_{\text{ess}}(h_\alpha(K-p))+\varepsilon_\alpha(p)\}.$$

According to Lemmas 2.2 and 2.3, and the relations $\varepsilon_1(p)=\varepsilon_2(p)=0$ and $\varepsilon_3(p)=\varepsilon(p)/m_3$, we obtain

$$\sigma_{\text{two}}(H_\alpha(K))=\begin{cases}\varnothing, & if \quad 0<\mu_\alpha\le\mu_0\\ \{z_\alpha\}, & if \quad \mu_\alpha>\mu_0\end{cases},\quad \alpha=1,2$$

$$\sigma_{\text{two}}(H_3(K))=\left[-\mu_3,\frac{6}{m_3}-\mu_3\right].$$

and

$$\sigma_{\text{three}}(H_\alpha(K))=\left[0,\frac{6}{m_3}\right],\quad \alpha=1,2,3.$$

These imply the following lemma.

Lemma 2.4

For every $K\in\mathbb{T}^3$, the following relations hold:

$$\sigma(H_\alpha(K))=\begin{cases}\left[0,\frac{6}{m_3}\right], & if \quad 0<\mu_\alpha\le\mu_0\\ \{z_\alpha\}\cup\left[0,\frac{6}{m_3}\right], & if \quad \mu_\alpha>\mu_0\end{cases},\quad \alpha=1,2,$$

$$\sigma(H_3(K))=\left[-\mu_3,\frac{6}{m_3}-\mu_3\right]\cup\left[0,\frac{6}{m_3}\right],$$

where $z_\alpha\in(-\infty,0)$, $\alpha=1,2$ is an eigenvalue of $h_\alpha(k)$, $\alpha=1,2$.

III. The essential spectrum of $\boldsymbol{H(K)}$

One of the remarkable results in the spectral theory of multi-particle continuous Schrödinger operators is the description of the essential spectrum of the Schrödinger operators in terms of cluster operators (the HVZ-theorem. See Refs. [20, 21] for the discrete case and [25] for a pseudo-relativistic operator).

Lemma 3.1 *For every $K \in \mathbb{T}^3$, the essential spectrum $\sigma_{ess}(H(K))$ of $H(K)$ is the union of the spectra of the channel operators $H_\alpha(K) = H_0(K) - \mu_\alpha V_\alpha$, $\alpha = 1,2,3$, i.e.,*

$$\sigma_{ess}(H(K)) = \sigma(H_1(K)) \cup \sigma(H_2(K)) \cup \sigma(H_3(K)).$$

The proof can be found in Refs. [20] and [21].

Therefore, according to Lemma 2.4, the structure of the essential spectrum of the operator $H(K)$ can be described by the following theorem.

Theorem 3.2 *Let $\mu_\alpha > 0$, $\alpha = 1,2$, and $\mu_3 \geq 0$. For the essential spectrum of the main operator $H(K)$, we have*

$$\sigma_{ess}(H(K)) = \left[0, \frac{6}{m_3}\right] \cup \left(\Lambda_1 \cup \Lambda_2 \cup \left[-\mu_3, \frac{6}{m_3} - \mu_3\right]\right),$$

where

$$\Lambda_\alpha = \begin{cases} \varnothing, & \text{if} \quad 0 \leq \mu_\alpha \leq \mu_0, \\ \{z_\alpha^0\}, & \text{if} \quad \mu_\alpha > \mu_0, \end{cases} \qquad \alpha = 1,2.$$

Particularly, we have

$$\sigma_{ess}(H(K)) = \Lambda_1 \cup \Lambda_2 \cup \left[-\mu_3, \frac{6}{m_3}\right], \quad \text{if} \quad 0 \leq \mu_3 \leq \frac{6}{m_3},$$

$$\sigma_{ess}(H(K)) = \Lambda_1 \cup \Lambda_2 \cup \left[-\mu_3, \frac{6}{m_3} - \mu_3\right] \cup \left[0, \frac{6}{m_3}\right], \quad \text{if} \quad \mu_3 > \frac{6}{m_3}.$$

IV. Conclusions

We studied the parameter-dependent discrete Hamiltonian corresponding to the system of two heavy and one light particles of arbitrary nature (either bosons or fermions) on $\mathbb{Z}^3$. The interactions between the particles were described via pairwise zero-range (contact) attractive forces. Using the direct-integral decomposition method, the three-particle problem was reduced to the study of simpler two-particle Schrödinger operators, called the channel (or fiber) operators. Using the channel operators, we found that the essential spectrum consists of a finite union of real segments.

Acknowledgements

The authors acknowledge support from the Innovative Development Agency of the Republic of Uzbekistan (Grant No. FZ–20200929224).

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