DOI: 10.5281/zenodo.15659514
Link: https://zenodo.org/records/15659514

ON THE EIGENVALUES OF THE DISCRETE LAPLACE OPERATOR ON COMBINATORIAL GRAPHS

Shukhrat Alladustov,
Department of "Higher and Applied Mathematics",
Tashkent State University of Economics,
E-mail: sh.alladustov@gmail.com

Abstract. *We study discrete and metric graphs and some of their properties. We define a Laplace operator acting on a graph as a difference operator and investigate its spectral properties. Moreover, we learn its eigenvalues and eigenvectors that represent the stationary states (wavefunctions) of the considered system. A relation between the number of connected components of a graph and multiplicity of* 0 *as an eigenvalue of the corresponding Laplace operator is established in examples.*

Keywords: *Graphs, connected components, Laplace operator, spectrum, eigenvalue.*

I. INTRODUCTION

The study of the spectral properties of Laplacian operators on discrete graphs are of interest due to their wide applications in physics and chemistry (see Refs. [1,2] for more details). Spectrum of discrete Laplacians is well-studied both for finite and infinite graphs (see [3,4,5] and references therein). Inverse spectral theory for the discrete Schrödinger operators with finitely supported potentials on the hexagonal lattice was studied by Ando [6]. Discrete graphs and their properties were also studied in the works [7-11].

In this work, we introduce some basic concepts of graphs, their types and properties. We also define Laplacian operator on graphs and study its spectral properties in examples.

The paper is organized as follows. Section I is the introduction. In section II, we describe discrete graphs and some basic concepts. Discrete Laplacian operators and their properties are discussed in section III. Eigenvalues of Laplacians are studied in examples. In Section IV, metric graphs are briefly discussed.

II. A DISCRETE GRAPH

A graph $\boldsymbol{\Gamma}$ consists of a finite or countably infinite set of vertices $\boldsymbol{\mathcal{V}} = \{v_i\}$ and a set $\boldsymbol{\mathcal{E}} = \{e_j\}$ of edges connecting the vertices. For example, the following is a graph with five vertices $\{v_i, i = 1, .., 5\}$ and four edges $\{e_i, i = 1, \dots, 4\}$ (see also [11]).

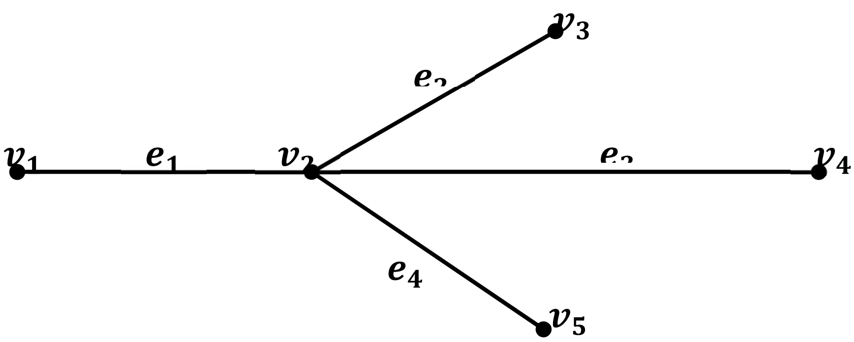


Figure 1. A simple graph

We assume that the edges of the graphs considered are undirected. Additionally, we require absence of loops and multiple edges, i.e., we consider simple graphs. We use the notations $E := |\mathcal{E}|$ and $V := |\mathcal{V}|$ for the number of edges and vertices, respectively. We use the notation $v \in e$ to mean that v is a vertex of the edge e. Two vertices u and v are called adjacent if they are connected by an edge. For adjacent vertices we use the notation $u \sim v$. In the above example, we have $v_1 \sim v_2$, $v_2 \sim v_3$, $v_2 \sim v_4$ and $v_2 \sim v_5$. Every simple graph can be specified by its adjacency matrix of dimension $|\mathcal{V}| \times |\mathcal{V}|$, defined as

$$A_{u,v} = \begin{cases} 1 & \text{if } u \sim v\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We note that with this definition adjacency matrix is symmetric. For the graph, demonstrated in Fig. 1, the adjacency matrix is of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $v \in \mathcal{V}$ be a vertex of a graph. Then the number of edges connecting v is called its degree and denoted as d_v.

***d*-regular graphs.** If degrees of every vertex of a graph are equal to d, then it is called a d-regular graph. For a d-regular map we have

$$|\mathcal{E}| = |\mathcal{V}| \cdot \frac{d}{2}$$

In the example below, presented is a 3-regular graph, with 4 vertices and 6 edges.

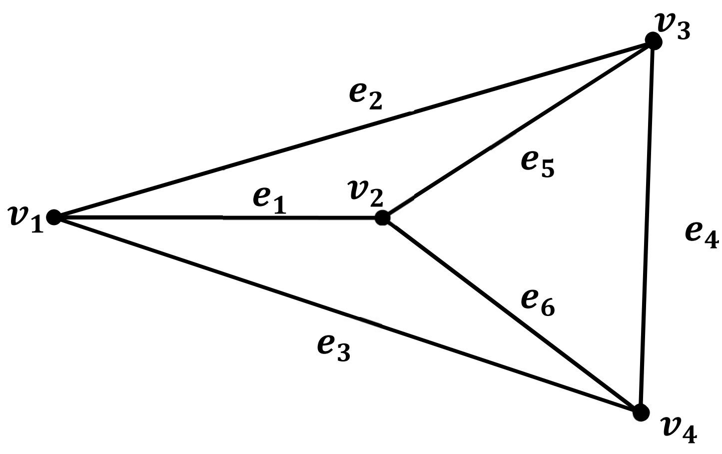


Figure 2. A 3-regular graph

Indeed, we have $|\mathcal{E}| = 6$ and $|\mathcal{V}| \cdot \frac{d}{2} = 4 \cdot \frac{3}{2} = 6$.

A **path** is a sequence of vertices $v_1, v_2, \dots, v_n$, such that each consecutive pair of vertices is connected by an edge. A **cycle** is a path that starts and ends at the same vertex.

A **tree** is a graph without cycles. A graph in Fig. 1 is a tree as it has no cycles, whereas the following is not a tree as it has a cycle $v_1 - v_2 - v_3$.

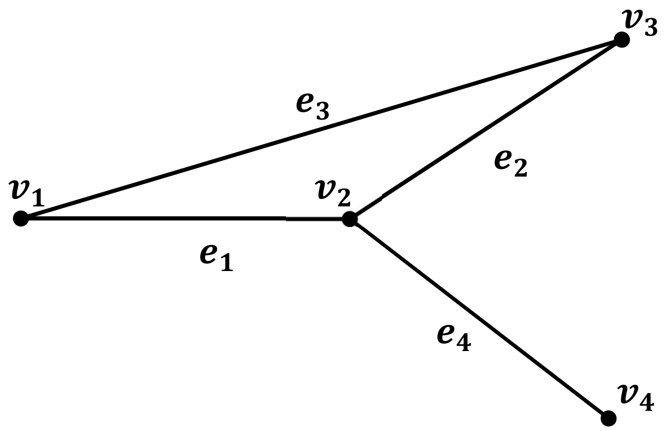


Figure 3. A simple graph which is not a tree

Let Γ be a connected graph, where any vertices can be connected by a path. Then **cyclomatic number** of a graph is the number of edges that have to be removed to turn the graph into a tree (see Ref. [6] for more details). For connected graphs, it is the same as 1st Betti number, denoted as β_Γ, and it is found as

$$\beta_\Gamma = |\mathcal{E}| - |\mathcal{V}| + 1$$

We note that $\beta_\Gamma = 0$ if and only if Γ is a tree.

Let us define functions defined on graphs. There are several ways of identifying a function on graphs, one of them is defining the function at the vertices. For example, a relation assigning the degree of an edge is a function from a graph to the set of integer numbers,

$$D: \mathcal{V} \to \mathbb{Z}, \qquad D(v) = d_v$$

III. DISCRETE LAPLACE OPERATOR

Let Γ be a discrete graph. The Laplace operator or Laplacian, L, acting on Γ is defined as

$$L = D - A$$

where D is the degree function and A is the adjacency function. Then,

$$L_{u,v} = \begin{cases} d_u & \text{if } u = v\,, \\ -1 & u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, it can be described as

$$Lf(u) = \sum_{v \in \mathcal{V}} L_{u,v} f(v) = d_u f(u) - \sum_{u \sim v} f(v)$$

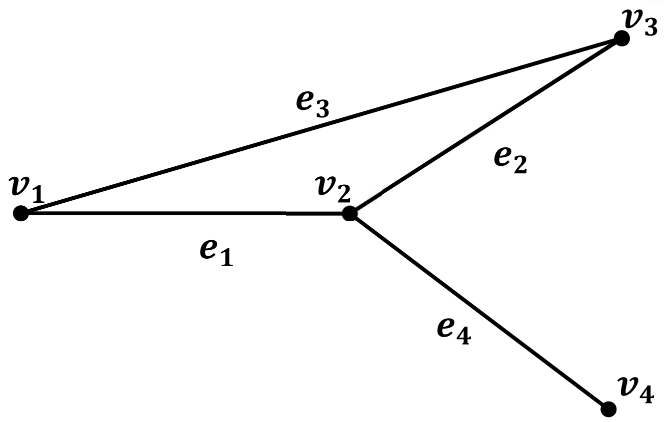
Theorem. Let Γ be a d-regular graph. Then 0 is an eigenvalue of the operator L.

Indeed, we can easily check that $D\begin{bmatrix}1\\1\\\cdots\\1\end{bmatrix}=\begin{bmatrix}d\\d\\\cdots\\d\end{bmatrix}$. Also, for the adjacency matrix, $A\begin{bmatrix}1\\1\\\cdots\\1\end{bmatrix}=\begin{bmatrix}d\\d\\\cdots\\d\end{bmatrix}$ as each row represents number of adjacent edges. Therefore, $L\begin{bmatrix}1\\1\\\cdots\\1\end{bmatrix}=0$.

A graph is called **connected** if any two of its vertices can be connected by a path. If a graph consists of several connected graphs, then connected graphs are called its **connected components**.

Theorem (Ref. [3]). A graph Γ has n connected components if and only if 0 is an eigenvalue of the Laplacian of multiplicity n. That is, if 0 is a simple eigenvalue, then it is connected and vice versa.

Example 1. Let us define the Laplace operator on the following graph



From the definition of the Laplace operator, we obtain that

$$L=\begin{bmatrix}2 & -1 & -1 & 0\\ -1 & 3 & -1 & -1\\ -1 & -1 & 2 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}$$

The spectrum of the Laplace operator on a graph gives important physical and mathematical information about the graph. Let us find its eigenvalues. For that we have to find the roots of the equation

$$\begin{vmatrix}2-\lambda & -1 & -1 & 0\\ -1 & 3-\lambda & -1 & -1\\ -1 & -1 & 2-\lambda & 0\\ 0 & -1 & 0 & 1-\lambda\end{vmatrix}=0$$

Simplifying the left side of the equation we obtain the characteristic function

$$\chi(\lambda)=\lambda^4-8\lambda^3+19\lambda^2-12\lambda$$

As

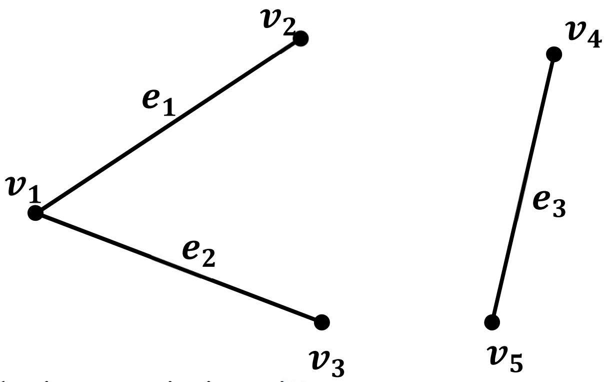
$$\chi(\lambda)=\lambda(\lambda^3-8\lambda^2+19\lambda-12)$$

and $\chi(1)=0$, we can factor it out as

$$\chi(\lambda)=\lambda(\lambda-1)(\lambda-4)(\lambda-3)$$

Therefore, eigenvalues of the Laplace operator are 0, 1, 3 and 4. As we can see that 0 is a simple eigenvalue, therefore it is connected.

Example 2. Let us now consider a graph with two connected components.



For this graph, Laplacian matrix is written as

$$L = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let us find its eigenvalues. We have

$$\begin{vmatrix} 2-\lambda & 1 & 1 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 & 0 \\ 1 & 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 0 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} \cdot \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

We can easily find that

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - 3\lambda$$

and

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda$$

Therefore, its characteristic form is

$$\chi(\lambda) = (-\lambda^3 + 4\lambda^2 - 3\lambda)(\lambda^2 - 2\lambda) = -\lambda^2(\lambda^2 - 4\lambda + 3)(\lambda - 2)$$

It can further be simplified as

$$\chi(\lambda) = -\lambda^2(\lambda - 3)(\lambda - 1)(\lambda - 2)$$

As we can see that that it has eigenvalues $0, 1, 2$ and 3. Moreover, we can see that 0 is an eigenvalue of multiplicity 2. So, we verified correctness of the theorem, as the graph has two connected components.

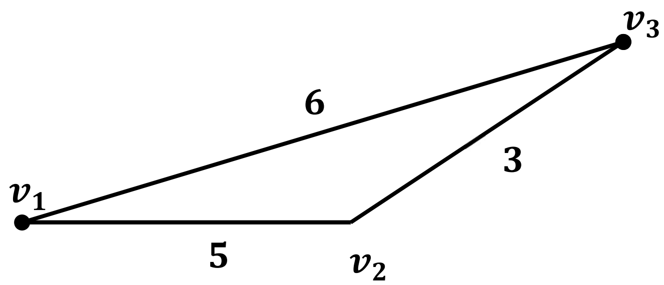
IV. METRIC GRAPHS

A metric graph G is a graph $\Gamma = \{\mathcal{E}, \mathcal{V}\}$, equipped with a metric l, i.e.,

$$G = \{\mathcal{E}, \mathcal{V}, l\}$$

where $l: \mathcal{E} \to \mathbb{R}^{>0}$ is a length function. Here l can be interpreted as physical distance or some other metric between the vertices. If an edge has only one vertex, then it has infinite length and it is called a lead. In the following graph, demonstrated three points with corresponding distances from each other. Then,

$$l(v_1, v_2) = 5$$
$$l(v_1, v_3) = 6$$
$$l(v_2, v_3) = 3$$



Length of a path is a sum of distances between the formed edges.

A metric graph is called **equilateral**, if distances between all vertices are equal.

A metric graph is called **infinite** if there are infinitely many vertices, otherwise it is called finite. A finite graph, where all edges have finite lengths is called a compact graph. In the above example, the graph is compact as the edges are all of finite lengths.

We can consider every metric graph as a metric space, where distance between any two vertices can be identified as a minimum length of the paths connecting those points.

We will continue studying metric graphs and quantum graphs.

V. REFERENCES

[1] Castro Neto, A. H., Guinea, F. Peres, N. M. R., Novoselov K.S., Geim A. The electronic properties of graphene, Rev. Mod. Phys. 81 (2009), 109–162.

[2] Harris, P., Carbon nano-tubes and related structure, Cambridge, Cambridge University Press, 2002.

[3] Berkolaiko, G., and Kuchment, P., Introduction to quantum graphs (Vol. 186). American Mathematical Society, (2013).

[4] Chung, F. Spectral graph theory, AMS, Providence, Rhode. Island, 1997.

[5] Korotyaev, E., and Saburova, N., Schrödinger operators on periodic discrete graphs. *Mathematical Physics*, 45(3), (2021) 123-145.

[6] Ando, K. Inverse scattering theory for discrete Schrödinger operators on the hexagonal lattice, Ann. Henri Poincare, 14 (2013), 347–383.

[7] Berge C., Graphs and hypergraphs. North-Holland Publishing Co., Amsterdam, North-Holland Mathematical Library, Vol. 6, (1973).

[8] West, D. B., Introduction to Graph Theory (2nd ed.). Pearson, (2001).

[9] Godsil, C., and Royle, G., Algebraic Graph Theory. Springer, (2001).

[10] Diestel, R., Graph Theory (5th ed.). Springer. (2017).

[11] Rosen, K. H., Discrete Mathematics and Its Applications (7th ed.). McGraw-Hill, (2012).